.

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Long large character sums

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Résumé

Cette thèse traite d'un sujet central de la théorie analytique des nombres, notamment celui des caractères de Dirichlet et plus particulièrememt, celui des sommes de caractères. Plus précisément, on y développe un résultat concernant la valeur maximale pouvant être atteinte par une longue somme de caractère. Chemin faisant, nous serons amenés à investiguer la structure de réseaux et nous en soutirerons un résultat intéressant.

Dans le Chapitre 1 sont discutées les notions et techniques nécessaires à l'élaboration de la preuve du résultat principal. On y discutera des notions d'analyse harmonique, de techniques classiques de théorique des nombres et l'on fera finalement un survol des nombres friables.

Le Chapitre 2 introduira la théorie propre aux caractères de Dirichlet et aux sommes de caractères. Les propriétés de bases et les théorèmes classiques seront couverts ainsi qu'un aperçu des résultats récents qui touchent de près au sujet principal de cette thèse.

On donnera au Chapitre 3 un premier résultat qui fera diverger la thèse dans le domaine des réseaux. Il s'agit d'un résultat auxiliaire au résultat principal, mais qui offre un intérêt indépendant aux sommes de caractères. Il sera question de l'ordre de grandeur des multiples d'un vecteur choisi dans un réseau, lorsque les multiplicateurs sont dans certaines classes de congruences.

Le Chapitre 4 servira de lien entre les réseaux et les caractères et on y appliquera les théorème démontrés au Chapitre 3. Les résultats sur les caractères qui en découlerons serons les éléments clés pour la preuve du théorème principal.

Au chapitre 5, nous dériverons quelques estimés préliminaires qui seront nécessaires à la preuve du théorème principal. En particulier, le chapitre sera divisé en deux sectioncs; l'une traitant de sommes exponentielles, l'autre de nombre friables.

Finalement, le Chapitre 6 constitura le point culminant de cette thèse et servira à démontrer le résultat principal sur les sommes de caractères. Nous y prouverons une borne inférieur sur le maximum pouvant être atteinte par un caractère parmi les caractères modulo un nombre premier q.

Mots-clés: Théorie de nombres analytique, Caractères de Dirichlet, Longues sommes de caractères, Réseaux.

Abstract

This thesis deals with a central topic in analytic number theory, namely that of characters and more specifically, that of character sums. More precisely, we will develop a result concerning the maximal value that can be attained by some long character sum.

In Chapter 1 are discussed the notions and techniques that will be necessary in the elaboration of the proof of the main result. We will discuss notions of harmonic analysis, classical number theoretic techniques, as well as give an overview of smooth numbers.

Chapter 2 will serve as an introduction to the theory pertaining to Dirichlet characters and character sums. Basic properties and classical theorems will be covered and we will provide a survey of recent results closely related to the main topic on interest in this thesis.

We will give in Chapter 3 a first result which will lead this thesis to diverge into the field of lattices. It comes up as an auxiliary result to the main result, but bares an interest independent to characters. We will discuss the order of magnitude of multiples of a chosen lattice vector, when the multipliers lie in prescribed congruence classes.

Chapter 4 will serve as a bridge between lattices and characters and we will study the consequences of applying the theorems we proved in Chapter 3 to characters. We will derive results that will be key to the proof of our main theorem.

In Chapter 5, we will prepare the ground for the proof of our main theorem by unveiling some preliminary estimates that will be needed. In particular, the chapter will consist of two parts: one treating of exponential sums, while the other one will be concerned with smooth numbers.

Finally, Chapter 6 will be the apex of this thesis and will provide the proof of our main result on character sums. The argument built in this chapter will allow us to prove a lower bound for the maximal value that can be reached by a character among the characters modulo a prime number q.

Keywords: Analytic number theory, Dirichlet characters, Long character sums, Lattices.

Contents

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Aux deux amours de ma vie, Édan et Aurélie.

Introduction

0.1. Overview of the problem

Arithmetic functions arise naturally in various ways in Number Theory, often as tools to describe interesting properties of integers or to extract useful information. Think for example of functions dissecting the anatomy of integers like $\omega(n)$, which counts the number of distinct prime divisors of n or to functions relating integers among each other like the Euler Totient function $\phi(n)$, which counts the number of previous integers coprime with n. Think also about functions analyzing the distribution of given sequences of integers, like the prime counting function $\pi(x)$ or the Dirichlet characters $\chi(n)$ from which we can gain information on the distribution of prime numbers in arithmetic progressions, for example.

As we undertake the task of studying arithmetic functions, we quickly have to face their irregular, sometimes chaotic behavior, which limits greatly our ability to understand the properties of the integers under consideration. In order to overcome this problem, we can draw inspiration from probabilistic questions and approach the problem in a broader way, asking for statistical quantities arising for arithmetic functions instead of trying to understand pointwise quantities. Among the questions of interest, we can ask about the density of some subsets of the integers, as well as the normal order and the extremal order of arithmetic functions. We will not develop on these here as this thesis is concerned with a different quantity, that of the mean value of an arithmetic functions, which often captures the essence of the function while smoothing major fluctuations. In general, it is difficult to get interesting results without having more specific information on the functions we wish to understand, and we are often compelled to study functions with more structure such as multiplicative functions, which obey the condition that $f(ab) = f(a)f(b)$ whenever $(a, b) = 1$. The function is said to be completely multiplicative if the coprimality condition can be ignored. In this thesis we will focus on the mean value of a specific type of completely multiplicative functions, the Dirichlet characters. As is the case for most arithmetic functions, there is limited information we can get on the values of a generic character, so we turn to the study of its mean value to help us extract information. That is, we are interested in understanding the growth of character sums of the form

$$\sum_{n\leq x}\chi(n), \tag{0.1.1}$$

where x is a positive real number and χ is a character modulo an integer q. More globally, we often wish to understand the extremal values of such sums, which leads us to study

$$\max_{x\leq q}\left|\sum_{\leq x}\chi(n)\right|. \tag{0.1.2}$$

Extensive research has been pursued on (0.1.1) and (0.1.2) over the last century and our knowledge on characters and character sums has been constantly growing since. One of the most well-known unconditional results providing an upper bound for (0.1.1) is undoubtedly the Pólya-Vinogradov inequality (1918).

Theorem. *[Pólya-Vinogradov inequality]*
For any positive real number x,

$$\sum_{n\leq x}\chi(n)\ll\sqrt{q}\log q.$$

No major improvements on the Pólya-Vinogradov inequality have come to light in the subsequent years, except perhaps the one obtained by Burgess in 1962, which roughly says that for $x\geq q^{\frac{1}{4}+\epsilon}$

$$\sum_{n\leq x}\chi(n)=o(x).$$

This result is still the state of the art nowadays, yet it is believed that the upper bound for (0.1.1) should be a little smaller than that of Pólya-Vinogradov. Indeed, if we believe the generalized Riemann hypothesis to be true, then by Montgomery and Vaughan's celebrated result, we should have that

$$\sum_{n\leq x}\chi(n)\ll\sqrt{q}\log\log q. \tag{0.1.3}$$

It was not until recently that the Pólya-Vinogradov inequality has been improved for some families of characters. Indeed, in [**8**], Granville and Soundararajan gave one of the first important improvement on character sums in a few decades. Their result was improved by Goldmakher [**5**] shortly after, as he showed

Theorem. *Let χ be a primitive character modulo q of fixed odd order g, then unconditionally*

$$\left|\sum_{n\leq x}\chi(n)\right| \ll_g \sqrt{q}(\log q)^{1-\delta_g+o_g(1)},$$

while under the generalized Riemann hypothesis, we have

$$\left|\sum_{n\leq x}\chi(n)\right| \ll_g \sqrt{q}(\log\log q)^{1-\delta_g+o_g(1)}, \tag{0.1.4}$$

where $\delta_g = 1-\frac{g}{\pi}\sin\left(\frac{\pi}{g}\right)$.

It's interesting to observe that under the generalized Riemann hypothesis, the second part of the theorem improves on (0.1.3) for characters of odd order. This is particularly surprising given that we know (see [**15**]) that there are infinitely many characters for which

$$\max_{x\leq q}\left|\sum_{n\leq x}\chi(n)\right| \geq \left(\frac{e^{\gamma}}{\pi}+o(1)\right)\sqrt{q}\log\log q. \tag{0.1.5}$$

It is therefore clear that any character χ satisfying (0.1.5) do not have bounded odd order. However, in this case, Goldmakher and Lamzouri [**6**] showed in a subsequent paper that (0.1.4) is best possible by proving

Theorem. *Let $g\geq 3$ be a fixed odd integer. There exist arbitrarily large q and primitive characters $\chi \pmod q$ of order g such that*

$$\left|\sum_{n\leq x}\chi(n)\right| \gg_{g,\epsilon} \sqrt{q}(\log\log q)^{1-\delta_g-\epsilon}, \tag{0.1.6}$$

where $\delta_g = 1-\frac{g}{\pi}\sin\left(\frac{\pi}{g}\right)$.

Results such as (0.1.5) and (0.1.6) are of great interest, as they allow one to quantify the accuracy of the upper bounds at hand. In that perspective, we often wish to derive omega results, that is to say, to establish lower bounds for character sums, or for some families of characters, and compare them with the known estimates in order to assess their quality. Ultimately, we would like to get insights on what should be the correct order of magnitude, which of course follows from having close upper and lower bounds. Among the developments in this direction, Granville and Soundararajan [**8**] showed the remarkable following result

Theorem. *Let q be a large prime and let $\theta \in (-\pi, \pi]$. Then there is an absolute constant C_0 such that for at least $q^{1-\frac{C_0}{(\log\log q)^2}}$ odd characters $\pmod q$ we have*

$$\sum_{n\le x} \chi(n) = e^{i\theta}\frac{e^{\gamma}}{\pi}\sqrt{q}\log\log q + O\left((\log\log q)^{\frac{1}{2}}\right)$$

for all but o(q) natural numbers $x \le q$.

Now, this means that there are a lot of characters for which

$$\max_{x\le q}\left|\sum_{n\le x}\chi(n)\right| \ge \left(\frac{e^{\gamma}}{\pi} + o(1)\right)\sqrt{q}\log\log q,$$

which is expected to be the correct order of magnitude for the maximal value reached by character sums.

Yet, although such results provide valuable insights on characters sums, they give no information as to where, in the interval $[1, q]$, these maximal values occur. This brings us to question the behavior of character sums when we do not ask for a generic bound that holds for all characters, but instead investigate the size of character sums for x in different ranges. That is, we would like to understand

$$\max_{\chi\ne\chi_0}\left|\sum_{n\le x}\chi(n)\right|, \tag{0.1.7}$$

where we may consider x in a specified range.

This is the question we will be interested in in this thesis and in particular, we will provide a lower bound for (0.1.7) when x is of the form $x = \frac{q}{(\log q)^B}$. Sums of the type (0.1.7) were first investigated by Granville and Soundararajan in their influential paper [**7**] published in 1999. In that paper, Granville and Soundararajan established lower bounds of (0.1.7) for x covering all ranges up to $\sqrt{q}$. (It actually goes beyond, but the result becomes essentially trivial past $\sqrt{q}$.) As an example, in the dual range from the one we cover in this thesis, we can deduce from one of the main results in their paper that

Theorem. *Let q be a large prime and suppose $x \leq (\log q)^B$. There are at least $q^{1-\frac{2}{\log x}}$ characters* $\pmod q$ *such that*

$$\max_{\chi \neq \chi_0} \left| \sum_{n \leq x} \chi(n) \right| \gg x\rho\left(\frac{\log x}{\log\log q}\right),$$

where $\rho(u)$ is the Dickman-De Bruijn function.

As we will see, it is interesting to note the resemblance between this result around $x = (\log q)^B$ and our result in Theorem 1 for the dual range $x = \frac{q}{(\log q)^B}$. On the other hand, in the range our main result is concerned with, their bound can be recovered via the mean square estimate, giving

$$\max_{\chi \neq \chi_0} \left| \sum_{n \leq \frac{q}{(\log q)^B}} \chi(n) \right| \gg \frac{\sqrt{q}}{(\log q)^{\frac{B}{2} - o(1)}}. \tag{0.1.8}$$

As far as we know, this was the best lower bound when the range of summation goes up to $x = \frac{q}{(\log q)^B}$. Comparing this with our theorem reveals that our result constitutes a major improvement to known lower bounds of the kind.

Indeed, as we would expect if we believe that the maximum obtained in (0.1.5) is reached around cq for a constant c, our result shows that thinking of B as being small, then for most prime moduli q the size of (0.1.7) is essentially of order $\sqrt{q}\log\log q$. In the next section, we make the precise statement of our theorem.

0.2. Statement of the results

In this thesis, we prove the following

Theorem 1. *Let Q be a large integer, for all but at most $Q^{\frac{1}{10}}$ primes $q \leq Q$, if $1 \leq B < \frac{\log\log\log q}{\log\log\log\log q}$, then*

$$\max_{\chi \neq \chi_0} \left| \sum_{n \leq \frac{q}{(\log q)^B}} \chi(n) \right| \geq \frac{\sqrt{q}}{\pi} \log\log q \int_B^\infty \rho(u)du + O(\sqrt{q}\log\log\log q).$$

We believe that Theorem 1 should hold for all prime moduli q, but we were unable to show this in the present thesis. Note however that the limitations come from the Fourier analysis argument used to prove Theorem 3 in Chapter 3, and we believe that improving this argument would make the bound in Theorem 1 hold for all prime moduli q.

What transpires from the proof is that the maximum in Theorem 1 is arising from odd characters. Hence, as a by-product of the proof of Theorem 1, singling out the case of even characters allows us to obtain a result that, although giving a weaker bound, holds for all prime moduli q.

Theorem 2. *Let q be a large prime and let $1 \leq B < \frac{\log\log\log q}{\log\log\log\log q}$, then*

$$\max_{\substack{\chi \neq \chi_0 \\ \chi \text{ even}}} \left| \sum_{n \leq \frac{q}{(\log q)^B}} \chi(n) \right| \geq \frac{\rho(B)}{2}\sqrt{q} + O\left(\frac{\rho(B)\log(B+1)\sqrt{q}(\log\log\log q)^2}{\log\log q} \right).$$

Again, observe the interesting appearance of the Dickman-De Bruijn ρ-function for smooth numbers in both our result and the dual result of Granville and Soundararajan (0.1.8). This is due to the fact that on both ranges, most of the contribution to the character sums comes from the small primes, so that we can essentially restrict the character sums to smooth numbers and thus the presence of $\rho(u)$.

Moreover, under the following conjecture, which we believe to be true, the inequalities in Theorem 1 and Theorem 2 become in fact equalities and therefore, we expect the results to be best possible.

Conjecture 1. *Let q be large and let χ be a non-principal character modulo q. Let $\log q \leq y \leq x$, then*

$$\max_{\alpha \in [0,1]} \left| \sum_{\substack{n \leq x \\ P(n) > y}} \frac{\chi(n)}{n} e(\alpha n) \right| \ll 1.$$

Now, although the size of the error term that stems from the proof of Theorem 1 limits B to range below $\frac{\log\log\log q}{\log\log\log\log q}$, we believe that the lower bound in Theorem 1 should extend to a wider range. In order for the bound to be non-trivial and be larger than the expected $\sqrt{x}$, we would roughly need $\sqrt{\frac{q}{(\log q)^B}} < \sqrt{q}B^{-B}$ and so we expect that we should be able to take B up to $\sqrt{\log q}$. Moreover, as we expect the large values of character sums to occur when $x = cq$ for some constant q, we think that the theorem should hold with $B > 0$. Hence, as an extension to Theorem 1 we conjecture

Conjecture 2. *Let q be a large prime and let $0 < B < \sqrt{\log q}$, then*

$$\max_{\chi \neq \chi_0} \left| \sum_{n \leq \frac{q}{(\log q)^B}} \chi(n) \right| \geq \frac{\sqrt{q}}{\pi} \log\log q \int_B^\infty \rho(u)du(1+o(1)).$$

Observe that this is consistent with known lower bounds such as (0.1.5), as letting B go to zero, and given that

$$\int_0^\infty \rho(u)du = e^\gamma,$$

we get,

$$\max_{\chi\neq\chi_0}\left|\sum_{n\leq\frac{q}{(\log q)^B}}\chi(n)\right| \geq \left(\frac{e^\gamma}{\pi}+o(1)\right)\sqrt{q}\log\log q$$

as expected.

Another by-product of our proof is an interesting result about lattices. The crux of the proof of Theorem 1 resides in finding an odd character pretending to be 1 on small primes and we will be required to do an incursion in the world of lattices in order to circumvent the difficulties encountered. Our quest about lattices will be about answering the following question:

Question. *Given a lattice vector* $\mathbf{u} = \frac{1}{M}(u_1,\cdots,u_k) \in (\mathbb{R}/\mathbb{Z})^k$ *of order* M*, is there an integer* $1 \leq \ell \leq M-1$ *such that all components of* $l\mathbf{u} \pmod 1$ *are small?*

Now, giving a positive answer is not difficult as a simple pigeonhole principle argument allows us to find many such vectors $\ell\mathbf{u}$. However, it could be that all such ℓ are of the form $\ell \equiv 0 \mod n$, which would restrict its applicability. Define

$$C_{n^+}(\eta,k) = \{0 \leq \ell \leq M-1\ ,\ \ell \equiv 0 \pmod n : |x_{\ell,j}| \leq \eta\ \forall 1 \leq j \leq k\}\,, \tag{0.2.1}$$

and

$$C_{n^-}(\eta,k) = \{0 \leq \ell \leq M-1\ ,\ \ell \not\equiv 0 \pmod n : |x_{\ell,j}| \leq \eta\ \forall 1 \leq j \leq k\}\,, \tag{0.2.2}$$

where $0 \leq x_{\ell,j} < 1$ is the j^{th} component of the vector $\mathbf{x}_\ell \equiv \ell\mathbf{u} \pmod 1$. Hence, the sets above contain all the vector multiples $\ell\mathbf{u}$ with small components. The pigeonhole principle gives the following:

Proposition 1. *Let* N *be a positive real number and let* $\mathbf{u} \in (\mathbb{R}/\mathbb{Z})^k$ *be a* k*-dimensional lattice vector of order* M*. Then for any fixed integer* $n < \frac{M}{N^k}$*,*

$$\#C_{n^+}\left(\frac{1}{N},k\right) \geq \frac{M}{nN^k}.$$

If, however, we wish to restrict our search to multiples of $\mathbf{x} \equiv \ell\mathbf{u}$ for $\ell \not\equiv 0 \mod n$, as we will with $n = 2$ in order to obtain Theorem 1, things get a little more tricky. Solving this problem gives rise to an interesting result about lattices, opening the door to further questions and conjectures in that vein. As this is of independent interest, the following theorems will be treated separately in Chapter 3. Using the usual Euclidean norm on $\mathbb{Z}^k$ and $(\mathbb{R}/\mathbb{Z})^k$, we get the two following theorems.

Theorem 3. *Let $N > 0$, k be a large integer and let $\mathbf{u} \in (\mathbb{R}/\mathbb{Z})^k$ be a lattice vector of order M. Given a divisor n of M then either*

(i) There exists a non-zero vector $\mathbf{r} \in (\mathbb{R}/\mathbb{Z})^k$ such that $|r_j| \le k^4 N \log^2(N)$ for $j \le k$ and $n(\mathbf{r}\cdot\mathbf{u}) \equiv 0 \pmod 1$,

or

(ii)

$$\#C_{n^-}\left(\frac{2}{N}, k\right) \ge \frac{M}{nN^k}.$$

Observe the light difference between Proposition 1 and Theorem 3, where in the first case we can choose n to be any integer and in the second case, we need the extra condition that n is a divisor of M.

Now, although we cannot quite show the converse, in the opposite direction, we have

Theorem 4. *Let $\mathbf{u} = \frac{1}{M}(u_1, \cdots, u_k) \in (\mathbb{R}/\mathbb{Z}^k)$ be a k-dimensional lattice vector of order M and let n be a divisor of M. Suppose that there exists $\mathbf{r} \in \mathbb{Z}^k$ such that $\mathbf{r}\cdot\mathbf{u} \equiv \frac{t}{n} \pmod 1$, where $(t, n) = 1$. Then for any integer $1 \le \ell \le M-1$ such that $\ell \not\equiv 0 \pmod n$, the vector $\mathbf{x} = \ell\mathbf{u} \in (\mathbb{R}/\mathbb{Z})^k$ satisfies*

$$|\mathbf{r}\cdot\mathbf{x} \pmod 1| \ge \frac{1}{n}. \tag{0.2.3}$$

In particular, if $|\mathbf{r}| \le L$, then

$$|\mathbf{x}| \ge \frac{1}{nL}.$$

We are not aware of any other result of the kind concerning lattices, and as such, we think that Theorems 3 and 4 are worth emphasizing. It would be interesting to push these ideas further and continue investigating in that direction to see if we could extract more information on lattices and their structure.

0.3. Applications of Theorem 1

Before diving into the heart of this thesis, we wish to mention that interesting ideas and results can emerge from Theorem 1. As a quick application, we highlight that even though it may be impossible to find very large even character sums for the range $x \le \frac{q}{(\log q)^B}$, there are intervals for which a bound such as the one in Theorem 1 should hold for even characters.

Indeed, for example, let ξ be an odd character $\pmod q$ for which the bound in Theorem 1 holds and let $\chi(n) = \xi(n)\left(\frac{n}{3}\right)$, where $\left(\frac{n}{3}\right)$ is the Legendre symbol $\pmod 3$. Observe that this means that $\chi(n)$ is an even character and that $\xi(n) = \chi(n)\left(\frac{n}{3}\right)$. Now let $x = \frac{q}{3(\log q)^B}$ and using $\overline{\chi}(3)\chi(3) = 1$, consider

$$\sum_{\frac{q}{3}\le n\le \frac{q}{3}+x} \chi(n) = \overline{\chi}(3) \sum_{\frac{q}{3}\le n\le \frac{q}{3}+x} \chi(3n).$$

Set $m = 3n - q$ and observe that in that case $\chi(3n) = \chi(m)$. That is, we have that

$$\begin{aligned}\sum_{\frac{q}{3}\le n\le \frac{q}{3}+x} \chi(n) &= \overline{\chi}(3) \sum_{\substack{m\le 3x\\ m\equiv -q \pmod 3}} \chi(m)\\
&= \frac{\overline{\chi}(3)}{2}\sum_{m\le 3x}\chi(m)\left(\psi_0(m) + \left(\frac{-q}{3}\right)\left(\frac{m}{3}\right)\right)\\
&= \frac{\overline{\chi}(3)}{2}\left(\sum_{m\le 3x}\chi(m)\psi_0(m) + \left(\frac{-q}{3}\right)\sum_{m\le \frac{q}{(\log q)^B}}\chi(m)\left(\frac{m}{3}\right)\right)\\
&= \frac{\overline{\chi}(3)}{2}\left(\sum_{m\le \frac{q}{(\log q)^B}}\chi(m)\psi_0(m) + \left(\frac{-q}{3}\right)\sum_{m\le 3x}\xi(m)\right),\end{aligned}$$

where ψ_0 is the principal character $\pmod 3$.

Now, $\chi(n)\psi_0(n)$ is even, so under the assumption that Theorem 2 is best possible then the first sum is small, and using Theorem 1 for the second sum, we get that

$$\left|\sum_{\frac{q}{3}\le n\le \frac{q}{3}+x} \chi(n)\right| = \frac{\sqrt{q}}{2\pi}\log\log q\int_B^\infty \rho(u)du + O(\sqrt{q}\log\log\log q).$$

Hence, this shows that there are even characters for which the character sum gets as large as in the odd character case, given that we perform an appropriate shift of the range. We have not pushed these ideas any further in this thesis, but it is likely that this line of thought would be fruitful in producing bounds for different ranges and restricted sets of integers.

We now describe briefly the way this thesis is organized.

0.4. Organization of the thesis

This thesis is organized as follows; in Chapter 1, we cover the preliminaries that will be necessary to the proof of our results, going through notions of harmonic analysis, number theory estimates and smooth numbers.

Chapter 2 will serve as a motivation for the problem we care about in this thesis. It will contain background about characters and character sums, as well as give a brief exposition on the work that has been achieved on the topic.

In Chapter 3, we will go off road and make an incursion in the world of lattices. That is, we will unveil the proofs of Theorem 3 and 4, which will be useful to prove our main theorem.

We will use Chapter 4 to apply our results on lattices to characters and study some of the consequences ensuing from it.

In Chapter 5, we will establish some technical estimates on exponential sums and smooth numbers that will be needed in the proof of Theorem 1.

Finally, Chapter 6 will be the culminating point of this thesis and will provide the proofs for Theorems 1 and 2.

Chapter 6 will be split in several sections, each containing various estimates. As we will see, the proof behind Theorem 1 relies on using Pólya's Fourier expansion for character sums

$$\sum_{n\leq x}\chi(n) = \frac{\tau(\chi)}{2\pi i}\sum_{\substack{|n|\leq z\\ n\neq 0}}\frac{\overline{\chi}(n)}{n}\left(1-e\left(\frac{-nx}{q}\right)\right)+O\left(1+\frac{q\log q}{z}\right), \tag{0.4.1}$$

and estimating (0.4.1) will require us to investigate separately different ranges. This will allow us to single out the crucial range, which, for $x=\frac{q}{(\log q)^B}$, is around $(\log q)^B$. From that crucial range and the different tools derived in the previous chapters, we will be able to obtain the main contribution leading to Theorem 1. Bonne lecture!

Chapter 1

Preliminary notions and techniques

Before diving into the heart of our problem, we start by covering the different techniques and notions that will be of use to us. In this chapter, we first discuss some of the notions that are pertaining to the proof of our main theorem and we give some background about smooth numbers that will be needed in the process.

1.1. Harmonic analysis

Harmonic analysis provides invaluable and powerful tools when studying functions. Starting from the idea of approximating or representing periodic functions as the sum of simpler waves through Fourier series, a rich field grew, providing means to understand functions by bridging the "time domain" of a function with its "frequency domain". For functions in $L^1(\mathbb{R})$, the space of integrable functions on $\mathbb{R}$, a function is represented on its frequency domain via its Fourier transform, which can be seen as the analogue of the Fourier series when we let the period go to infinity. This bridge between a function and its Fourier transform allows us to handle problems that could be hard to tackle on one of the two sides. Indeed, by transferring the questions from one side to the other, we sometimes have access to better tools to work with. In the proof of Theorem 1, harmonic analysis will play a crucial role and this section lays out the important features of the Fourier transform that will allow us to build the argument later on, in Theorem 3. Most of the exposition in this section will be done for the Fourier transform in 1 dimension for simplicity, but the extension to higher dimension happens quite naturally and we will discuss it briefly at the end. We start with some definitions and proceed to discuss some important properties of the Fourier transform. In the following, let $L^1 = L^1(\mathbb{R})$ be the space of integrable functions over $\mathbb{R}$.

Definition 1.1.1. *Let $f \in L^1$ be an integrable function, and denote $\mathcal{F}$ for the Fourier transform operator, then the Fourier transform of f is defined by*

$$\mathcal{F}(f)(y) = \hat{f}(y) = \int_{\infty}^{\infty} f(x)e(-xy)dx,$$

where e(x):= $e^{2\pi i x}$.

This operation on f is reversible, creating Fourier pairs intimately related to each other. First, define

Definition 1.1.2. *Let* $\mathcal{F}^{-1}$ *be the inverse Fourier transform operator. Then for any* $g \in L^1$, *we define the inverse Fourier transform as*

$$\mathcal{F}^{-1}(g)(x) = \int_{-\infty}^{\infty} g(y)e(xy)dy.$$

In general, for a function $f \in L^1$, we have that

$$\int_{-Y}^{Y} \left(1 - \frac{|y|}{Y}\right) \hat{f}(y)e(xy)dy \to f(x)$$

as $Y \to \infty$ in the L^1-norm. If we also have $\hat{f} \in L^1(\mathbb{R})$, then the integral converges uniformly and we get that

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(y)e(xy)dy,$$

forming a unique pair of functions f and $\hat{f}$ under the Fourier transform.

Many of the properties of the Fourier transform can be used to our advantage when studying functions. We state some of its important properties, many of which will be used in the proof of Theorem 3. We start with a few basic properties.

Proposition 1.1.1. *(Properties of the Fourier transform)*
Let f *be a function in* L^1, *let* $\hat{f}$ *be its Fourier transform and let* $\mathcal{F}$ *be the operator such that* $\mathcal{F}(f) = \hat{f}$.

(1) ***Linearity***

$$\mathcal{F}(af + bg) = a\hat{f} + b\hat{g}$$

(2) ***Translation***

$$\mathcal{F}(f(x-a))(y) = e(-ay)\hat{f}(y)$$

(3) ***Time scaling***

$$\mathcal{F}(f(ax))(y) = \frac{1}{|a|}\hat{f}\left(\frac{y}{a}\right)$$

(4) ***Differentiation***

$$\mathcal{F}(f'(x))(y) = 2\pi i y \hat{f}(y)$$

(5) L^1 ***bound***

$$|\hat{f}(y)| \leq \int_{\mathbb{R}} |f(x)| dx < \infty.$$

One of the key features of the Fourier transform resides in its fast rate of decay, which gets better as f gets smoother. That is, the higher derivatives the function has, the faster its Fourier transform decays. More precisely, we have

Theorem 1.1.1. *Suppose* $f, f', \ldots, f^{(n)} \in L^1(\mathbb{R})$, *then*

$$\hat{f}(y) \ll \frac{1}{1+|y|^n}.$$

In particular, if $f \in C^\infty$, *then for every* $k \geq 0$

$$\hat{f}(y) \ll \frac{1}{1+|y|^k}.$$

One point of interest of Theorem 1.1.1 lies in the fact that, because of the fast decay properties of functions f with derivatives in L^1, one can in practice essentially focus on y with small modulus to gain most of the information on $\hat{f}$. In the context of our proof, this will enable us to perform a truncation to one of the key summations performed over values of a Fourier transform.

In order to take full advantage of this feature of the Fourier transform, one would ideally like to work with functions in C^∞. However, it regularly happens that the functions f under study do not belong to C^∞. One way to circumvent this problematic is to smoothen (or mollify) the function by taking the convolution of f with an infinitely smooth function (or even better a Schwartz function).

Definition 1.1.3. *The convolution of two functions* f *and* g *is defined as*

$$f \star g(x) = \int_{-\infty}^{\infty} f(t) g(x-t) dt.$$

The convolution operation can be very profitable, especially when working with the Fourier transform, as it transfers the smooth properties of g to $f \star g$. Of course, the downside is that we no longer work with the original function f, although one usually tries to choose a smooth (Schwartz) function g such that $f \star g$ approximates f in a decent manner. A good choice for such functions is often given by the so-called bump functions. For example, in order to smoothen the indicator function for the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$

$$\mathbf{1}_{\left[-\frac{1}{2},\frac{1}{2}\right]}(x) = \begin{cases} 1 & \text{if } x \in \left[-\frac{1}{2}, \frac{1}{2}\right] \\ 0 & \text{otherwise,} \end{cases}$$

we will choose a bump function of the form

$$\phi(x) = \begin{cases} ce^{-\frac{1}{1-(2x)^2}} & \text{if } x \in \left[-\frac{1}{2}, \frac{1}{2}\right] \\ 0 & \text{otherwise,} \end{cases} \tag{1.1.1}$$

where c is a normalizing constant. Now, the convolution with $\phi(x)$ does not provide a very good approximation of $\mathbf{1}_{\left[-\frac{1}{2},\frac{1}{2}\right]}(x)$, but this can be improved by taking $\phi_\epsilon = \epsilon^{-1}\phi\left(\frac{x}{\epsilon}\right)$ which goes to the Dirac delta function $\delta(x)$ as $\epsilon \to 0$.

As we will see, the convolution $\mathbf{1}_{\left[-\frac{1}{2},\frac{1}{2}\right]} \star \phi_\epsilon(x)$ will serve a double purpose in our proof, allowing us to reach the result in Theorem 3. To understand how the convolution acts in the context of the Fourier transform for compactly supported functions, we now describe some of its properties and how it transfers through the Fourier transform.

Proposition 1.1.2. *(Properties of convolution)*
Let f and g be compactly supported functions in L^1 then

(1) ***Compact support***
$f \star g(x)$ is in L^1 and also has compact support.

(2) ***Algebra***
The space of compactly supported functions in L^1 endowed with the convolution property forms an algebra with identity.

(3) ***Identity***
The Dirac $\delta(x)$ function acts as the identity element with respect to convolution. That is

$$f \star \delta(x) = f(x)$$

.

(4) ***Differentiation***

$$(f \star g)'(x) = f' \star g(x) = f \star g'(x).$$

By the differentiation property, we get the following

Corollary 1.1.1. *Suppose that $f \in L^1$ and that $g \in C^n$, then $f \star g \in C^n$.*

Although we can apply the Fourier transform to any function in L^1, it is often fruitful to restrict to a special class of integrable functions for which additional useful properties can be derived. Simply put

Definition 1.1.4. *A function f is said to be a Schwartz function if it is in C^∞ and it satisfies $|f(x)| \ll \frac{1}{|x|^n}$ as $x \to \infty$ for all n.*

In other words, the Schwartz space is the space containing functions which are decreasing really fast. Observe that our choice of bump functions (1.1.1) is in the Schwartz space, which will be very convenient for us.

Corollary 1.1.2. *Let $f \in L^1$ and suppose g is a Schwartz function, then so is $f \star g$. In particular, the space of Schwartz functions is closed under convolution.*
Moreover, the Fourier transform preserves the space of Schwartz functions.

By the properties of integrals, convolution transfers very nicely under the Fourier transform, as shows the convolution theorem.

Theorem 1.1.2. *(Convolution theorem)*
Let f and g be function in L^1, then

$$\mathcal{F}(f \star g) = \mathcal{F}(f)\mathcal{F}(g).$$

Hence, we see that the convolution of two functions transfers to the product of their associated functions under the Fourier transform. Notice that in particular, this confirms that if either f of g is infinitely smooth, then the convolution is also bounded by $\widehat{f \star g}(y) \ll \frac{1}{y^n}$ for any power n.

Now, when working with functions with fast decay like the Schwartz functions, we have access to some additional powerful tools allowing us to analyze functions that would otherwise be out of reach. In the proof of Theorem 3, we will need to following classical result relating a function and its Fourier transform.

Theorem 1.1.3 (Poisson summation formula)**.** *Let f be a Schwartz function, then*

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{k\in\mathbb{Z}} \hat{f}(k).$$

Proof. Let f be a Schwartz function and define

$$F(x) = \sum_{m\in\mathbb{Z}} f(x+m).$$

Then $F(x)$ is periodic of period 1 and therefore, we can take its Fourier expansion

$$F(x) = \sum_{k\in\mathbb{Z}} a_k e(kx),$$

where the coefficients are given by

$$a_k = \int_0^1 F(x)e(-kx)dx.$$

Now, substituting $F(x)$ by its definition,

$$\begin{aligned} a_k &= \int_0^1 \sum_{n\in\mathbb{Z}} f(x+n)e(-kx)dx \\ &= \sum_{n\in\mathbb{Z}} \int_0^1 f(x+n)e(-kx)dx \\ &= \sum_{n\in\mathbb{Z}} \int_n^{n+1} f(x)e(-kx)dx \\ &= \int_{\mathbb{R}} f(x)e(-kx)dx \\ &= \hat{f}(k). \end{aligned}$$

Thus, it follows that

$$F(x) = \sum_{k\in\mathbb{Z}} \hat{f}(k)e(kx),$$

and choosing $x=0$, we indeed get

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{k\in\mathbb{Z}} \hat{f}(k).$$

□

The convergence theorems of the Fourier transform are very useful when dealing with Poisson Formula. This allows one to truncate the summation and work with a short sum of Fourier transforms. Although we could state numerous useful results, we restrict ourselves to a result from [**13**] that will be used in Chapter 2 to truncate Pólya's Fourier expansion for character sums.

Theorem 1.1.4. *If $f(x)$ has bounded variation, then for any $\alpha \in \mathbb{R}$ where f is continuous,*

$$\left| f(\alpha) - \sum_{|k| \leq K} \hat{f}(k) e(\alpha k) \right| \leq \int_{0^+}^{1^1} \min \left\{ \frac{1}{2}, \frac{1}{(2K+1)\pi \sin(\pi x)} \right\} |df(\alpha + x)|.$$

We end this section by making some comments on how the theory discussed above extends to the n-dimensional case. So let $f \in L^1(\mathbb{R}^n)$, and define the n-dimensional Fourier transform by

Definition 1.1.5.

$$\hat{f}(\mathbf{y}) = \int_{\mathbb{R}^n} f(\mathbf{x}) e(\mathbf{x} \cdot \mathbf{y}) d\mathbf{x},$$

where bold variables denote n-dimensional vectors in $\mathbb{R}^n$ and $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \cdots + x_n y_n$ is the dot product between the vectors $\mathbf{x}$ and $\mathbf{y}$.

Now, this is a natural extension of the 1-dimensional case and all the basic properties of the Fourier transform transfer without too much effort. The approach we will favor in this work is the use of separable functions. This will make the treatment a lot easier, as for separable functions of the form

$$f(\mathbf{x}) = \prod_{k \leq n} f_k(x_k),$$

we can essentially reduce the problem to the 1-dimensional case. Indeed, by the properties of integrals, then

$$\hat{f}(\mathbf{y}) = \prod_{k \leq n} \hat{f}_k(y_k).$$

As we will not need any of the more involved features of n-dimensional Fourier transform, we won't enter the subject in more details here and instead, we will move on to the discussion of some techniques and estimates that are more specific to number theory and that will be needed to prove Theorem 1.

1.2. Number theoretic techniques

Being one the oldest branch in mathematics, there is something absolutely beautiful in the way number theory reconciles many areas of mathematics, be it by showing up in problems concerned with seemingly unrelated topics or by the way it borrows from many of the other fields to tackle problems about integers. Still, there are some techniques that can be attached mainly to number theory that emerged through the study of integers and related questions. In this section, we briefly cover simple tools and estimates that are regularly used in analytic number theory when studying arithmetic functions.

1.2.1. Partial summation

We start with a simple technique which allows us to turn problems about summations into a form in which we can use tools from calculus. As simple as this technique is, it will be our tool of predilection in the proof of Theorem 1, as it will enable us to use known estimates and calculus to handle our problem. So let a_n be a sequence of complex numbers and let f be a real-valued differentiable function over $\mathbb{C}$. Define also

$$S(t) = \sum_{n \leq t} a_n.$$

Then the following holds.

Proposition 1.2.1.

$$\sum_{A<n\leq B} a_n f(n) = S(B)f(B) - S(A)f(A) - \int_A^B S(t)f'(t)dt$$

Proof. Using the Riemann-Stieljes integral, we have

$$\sum_{A<n\leq B} a_n f(n) = \int_A^B f(t)d(S(t)),$$

and integration by parts give

$$\begin{aligned}\sum_{A<n\leq B} a_n f(n) &= [S(t)f(t)]_A^B - \int_A^B S(t)f'(t)dt \\ &= S(B)f(B) - S(A)f(A) - \int_A^B S(t)f'(t)dt\end{aligned}$$

as claimed. □

Notice that this is just a special case of the Riemann-Stieljes integral which does not actually ask for the functions to be differentiable. Hence, we make the remark that in the case where f is not differentiable, the same techniques can still prove to be very useful, in particular if $S(t)$ happens to have a good differentiable approximation. Through the proof of Theorem 1, we will use partial summation repetitively to handle sums in various contexts.

1.2.2. Some useful estimates: Mertens theorems

At the root of multiplicative number theory lie the prime numbers. Being the building blocks from which are constructed the integers, many questions concerning integers can be brought to understanding questions and estimates about primes. In the course of our proof, we will be required to use some classical estimates on prime numbers. Namely, we will use three theorems from Franz Mertens that go all the way back to 1874, the famous Mertens' theorems. As a historical note, observe that these theorems, which unveil information on the density of the prime numbers, came about 20 years before the celebrated Prime number Theorem of 1896, making them all the more interesting.

Our first estimate from the trilogy follows from a weak version of Stirling's formula and partial summation.

Proposition 1.2.2. *(Mertens' first theorem)*

$$\sum_{p \leq x} \frac{\log p}{p} = \log x + O(1).$$

Next, using partial summation, one can deduce that

Proposition 1.2.3. *(Mertens' second theorem) Let $x \geq 1$ be a positive real number, then*

$$\sum_{n \leq x} \frac{1}{p} = \log \log x + M + O\left(\frac{1}{\log x}\right),$$

where M is the Meissel-Mertens' constant.

Finally, using the properties of logarithms and Taylor series, we can obtain Merten's third theorem.

Proposition 1.2.4. *(Mertens' third theorem)*

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x} \left(1 + \left(\frac{1}{\log x}\right)\right),$$

where γ is Euler's constant.

As we will see, each of these theorems will be used in the proof of Theorem 1, the first two allowing us to bound some error terms, while the last one will be used to obtain the main term from the theorem.

1.3. Smooth numbers

Smooth numbers come up regularly when studying questions about multiplicative functions and also in solving problems in computational number theory. They have therefore been at the center of a lot of research in the last few decades, going back to 1930 with Dickman's remarkable result that the number of smooth numbers has non-zero density as we go to infinity. As they play a significant role in the proof of our main theorem, we devote this section to some definitions and results on smooth numbers that will be useful in the following sections.

Definition 1.3.1. *An integer is said to be y-smooth if all of its prime divisors p are such that $p \leq y$.*

Next we define the y-smooth counting function

$$\psi(x, y) := \sum_{\substack{n \leq x \\ p|n \implies p \leq y}} 1.$$

Closely related to smooth numbers is the Dickman-De Bruijn ρ-function, which is defined to be the unique function satisfying the delay-differential equation

$$u\rho'(u) = -\rho(u-1),$$

under the initial condition $\rho(u) = 1$ for $0 \leq u \leq 1$. Equivalently, with the same initial conditions, it can be defined as the unique solution to the integral-delay equation

$$u\rho(u) = \int_{u-1}^{u} \rho(t)dt.$$

We start by a classical estimate on smooth numbers, due to Hildebrand ([**16**] p.369), which exhibits the intimate relation between $\rho(u)$ and smooth numbers.

Lemma 1.3.1. *Let* $\exp((\log \log x)^{5/3+\epsilon}) \leq y$ *and let* $u = \frac{\log x}{\log y}$, *then*

$$\psi(x, y) = x\rho(u)\left(1 + O\left(\frac{\log(u+1)}{\log y}\right)\right).$$

By using Hildebrand's Theorem, we will be required to understand $\rho(u)$, so next we give some of its properties that will be useful to us. We first give an estimate for the size of $\rho(u)$, which follows from Lemma 3.1 in [**2**].

Lemma 1.3.2. *For $u \geq 1$,*

$$\rho(u) \ll u^{-u}.$$

The next two lemmas will allow us to deduce an estimate that we will use numerous times in order to prove Theorem 1. They appear as Corollary 8.3 and Lemma 8.1 in [**16**].

Lemma 1.3.3. *For any integer $k \geq 0$ and real number $u_1 > 1$, if $u \geq u_1$, then we have*

$$\rho^{(k)}(u) = (-1)^k \xi(u)^k \rho(u) \left(1 + O\left(\frac{1}{u}\right)\right),$$

where $\xi(u)$ is the unique real non-zero root of the equation $e^{\xi} = 1 + u\xi$.

Lemma 1.3.4. *For $u \geq 3$*

$$\xi(u) = \log(u \log u) + O\left(\frac{\log\log u}{\log u}\right).$$

Corollary 1.3.1. *Let* $\exp((\log\log x)^{5/3+\epsilon}) \leq y$ *and let $u = \frac{\log x}{\log y}$, then*

$$\psi(x,y) = x\rho(u) + O\left(\frac{x|\rho'(u)|}{\log y}\right).$$

Proof. From Lemmas 1.3.3 and 1.3.4, we have that

$$\rho'(u) = -\log(u \log u)\rho(u) + O\left(\frac{\rho(u)\log\log u}{\log u}\right),$$

which gives, together with Hildebrand's estimate

$$\psi(x,y) = x\rho(u) + O\left(\frac{x|\rho'(u)|}{\log y}\right).$$

□

Using Lemmas 1.3.3 and 1.3.4, we now show that moving away from u by a small amount does not affect too much the value of the ρ-function.

Lemma 1.3.5. *For $|v| \leq \frac{1}{\log u}$, we have*

$$\rho(u+v) = \rho(u)\left(1 + O(|v|\log(u+1)\right).$$

Proof. By the mean value theorem, there exists $u_0 \in [u, u+v)$ such that

$$\begin{aligned} \rho'(u_0) &= \frac{\rho(u+v) - \rho(u)}{(u+v) - u} \\ \implies \rho(u+v) &= \rho(u) + v\rho'(u_0). \end{aligned}$$

Now by Lemmas 1.3.3 and 1.3.4, we have that

$$\begin{aligned} \rho'(u_0) &= -\xi(u_0)\rho(u_0)\left(1 + O\left(\frac{1}{u_0}\right)\right) \\ &= O((\log(u+v)\rho(u)). \end{aligned}$$

It follows that

$$\rho(u+v) = \rho(u) + O(|v|\rho(u)\log(u+1)).$$

□

If we restrict v to be positive and we make a different use of the same lemmas, it is possible to take a much wider range and show

Lemma 1.3.6. *Let $u > 1$ and $0 < v \leq \frac{u}{\log u}$, then*

$$\rho(u+v) = \frac{\rho(u)}{u^{(1+o(1))v}},$$

as $u \to \infty$.

Proof.

$$\begin{aligned}
\log\left(\frac{\rho(u+v)}{\rho(u)}\right) &= \log(\rho(u+v)) - \log(\rho(u)) \\
&= \int_u^{u+v} (\log(\rho(t))' dt \\
&= \int_u^{u+v} \frac{\rho'(t)}{\rho(t)} dt
\end{aligned}$$

and using Lemmas 1.3.3 and 1.3.4 to write

$$\rho'(t) = \rho(t)\left(-\log(t\log t) + O\left(\frac{\log\log t}{\log t}\right)\right).$$

We get that

$$\begin{aligned}
\log\left(\frac{\rho(u+v)}{\rho(u)}\right) &= -\left(\int_u^{u+v} \log t dt + \int_u^{u+v} \log\log t dt\right) + O\left(\frac{v\log\log u}{\log u}\right) \\
&= -\left(\Big[t\log t - t\Big]_u^{u+v} + O(v\log\log(u+v)\right) + O\left(\frac{v\log\log u}{\log u}\right) \\
&= -\left(v\log(u+v) + O(v\log\log(u+v)\right) \\
&= -\left(v\log u + O\left(\frac{v}{\log u}\right) + O(v\log\log(u+v))\right) \\
&= -\left(1 + O\left(\frac{\log\log u}{u}\right)\right) v\log u.
\end{aligned}$$

Hence we can conclude that, given $v \leq \frac{u}{\log u}$, then

$$\frac{\rho(u+v)}{\rho(u)} = \exp(-(1+o(1))v\log u),$$

so that

$$\rho(u+v) = \frac{\rho(u)}{u^{(1+o(1))v}},$$

where we replaced $O\left(\frac{\log\log u}{\log u}\right)$ by $o(1)$ assuming that $u \to \infty$.

$\square$

The next lemma approximates the sum of reciprocals of y-smooth integers using the Dickman-De Bruijn's function. It follows directly from the strong version of Lemma 3.3 in [**2**]. (See remark 3.1)

Lemma 1.3.7. *Let $y \geq 2$ and $0 < s \leq r$, then*

$$\sum_{\substack{y^s \leq n \leq y^r \\ P(n) \leq y}} \frac{1}{n} = \log y \int_s^r \rho(t)dt + O(\rho(s)).$$

Proof. First, suppose that $s > 1$. Then using partial summation and Corollary 1.3.1, we have that

$$\begin{aligned}
\sum_{\substack{y^s < n \leq y^r \\ P(n) \leq y}} \frac{1}{n} &= \frac{1}{y^r}\psi(y^r, y) - \frac{1}{y^s}\psi(y^s, y) + \int_{y^s}^{y^r} \frac{\psi(t,y)}{t^2}dt \\
&= O(\rho(s)) + \int_{y^s}^{y^r} \frac{t\rho(u) + O\left(\frac{t|\rho'(u)|}{\log y}\right)}{t^2}dt \\
&= \int_{y^s}^{y^r} \frac{\rho(u) + O\left(\frac{|\rho'(u)|}{\log y}\right)}{t}dt.
\end{aligned}$$

Now changing variable $t = y^u$ leads to

$$\begin{aligned}
\sum_{\substack{y^s < n \leq y^r \\ P(n) \leq y}} \frac{1}{n} &= \log y \int_s^r \rho(u)du + O\left(\int_s^r |\rho'(u)|du\right) + O(\rho(s)) \\
&= \log y \int_s^r \rho(u)du + O(\rho(s)).
\end{aligned}$$

If $s \leq 1$, then for $s \leq u \leq 1$ we have that $\rho(u) = 1$ and therefore,

$$\begin{aligned}
\sum_{\substack{y^s < n \leq y^r \\ P(n) \leq y}} \frac{1}{n} &= \sum_{y^s < n \leq y} \frac{1}{n} + \sum_{\substack{y < n \leq y^r \\ P(n) \leq y}} \frac{1}{n} \\
&= (1-s)\log y + O(1) + \log y \int_1^r \rho(u)du + O(1) \\
&= \log y \int_s^1 \rho(u)du + \log y \int_1^r \rho(u)du + O(1) \\
&= \log y \int_s^r \rho(u)du + O(\rho(s)),
\end{aligned}$$

as $1 = \rho(s)$. This ends the proof of Lemma 1.3.7. $\square$

We end this section with a useful property of the Dickman-De Bruijn function which can be found in [**2**] Lemma 3.3.

Lemma 1.3.8. *Let γ be the Euler-Mascheroni constant, we have*

$$\int_0^\infty \rho(u)du = e^\gamma.$$

Now that we have set up the stage with various tools, we make our way to Chapter 2 and discuss the theory of characters, building the motivation behind the investigation of the problem undertook to solve in this thesis.

Chapter 2

Character sums

2.1. Dirichlet Characters

2.1.1. Basic properties

In this section, we give a brief exposition on the basic properties of our main object of study, Dirichlet characters. Let us start simply by defining what they are and then proceed to state the important properties they share, so that we can motivate the importance of understanding how they behave.

Definition 2.1.1. *A **Dirichlet character** modulo q is a homomorphism $\chi : \mathbb{Z}/q\mathbb{Z}^\times \to \mathbb{C}$ and as such, it respects the following conditions:*

(1) $\chi(nm) = \chi(n)\chi(m)$ for every $m, n \in \mathbb{Z}/q\mathbb{Z}^\times$

(2) $\chi(1) = 1$.

This is saying that Dirichlet characters are completely multiplicative, and as a direct consequence from the definition, if we let $k = \phi(q)$, then $\chi(m)^k = \chi(m^k) = \chi(1) = 1$, therefore $\chi(m)$ is a root of unity. This is a key feature of characters from which we draw the important orthogonality relation of characters. But let us start by extending our definition to all integers, by letting $\chi(a) = \chi(b)$ whenever $a \equiv b \pmod q$ and $\chi(n) = 0$ if $(n, q) > 1$, so that χ is a function on the integers into the complex numbers of period q, which takes values among the $\phi(q)^{\text{th}}$ roots of unity. Let us now state a few important definitions and properties of characters.

Definition 2.1.2. *A character χ modulo q is said to be **primitive** if whenever $\chi(n) = \chi(m)$ for all $m \equiv n \pmod d$, then we have that $d = q$.*

In other words, χ does not have periodicity smaller than q. Note that for simplicity, in this thesis, we will mostly be concerned with primitive characters.

Proposition 2.1.1 (Properties of characters)**.**

(1) The characters modulo q form a group G of order $\phi(q)$ with respect to multiplication. That is $\chi_1\chi_2(n) = \chi_1(n)\chi_2(n)$.
The identity element is called the principal character χ_0, with $\chi_0(n) = 1$ $\forall n \in \mathbb{Z}$, $(n,q) = 1$

(2) $G \simeq \mathbb{Z}/q\mathbb{Z}^{\times}$

(3) ***Orthogonality of characters***

$$\sum_{\substack{n=1 \\ (n,q)=1}}^{q} \chi(n) = \begin{cases} \phi(q) & \text{if } \chi = \chi_0 \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{\chi} \chi(n) = \begin{cases} \phi(q) & \text{if } n \equiv 1 \pmod{q} \\ 0 & \text{otherwise} \end{cases}$$

This last property bares the reason Dirichlet introduced characters in 1837. Indeed, endowed with their orthogonality property, characters act as a characteristic function for residue classes, which allowed Dirichlet to prove his famous theorem about primes in arithmetic progressions. More precisely, denoting $\overline{\chi}$ for the conjugate of χ, the orthogonality of characters implies that we can catch a prescribed residue class in the following manner

$$\frac{1}{\phi(q)} \sum_{\chi} \overline{\chi(a)}\chi(n) = \begin{cases} 1 & \text{if } n \equiv a \pmod{q} \\ 0 & \text{otherwise} \end{cases}.$$

Now, because characters are completely multiplicative, they are defined exclusively by the values they take on prime numbers and therefore, combined with their periodicity, they offer a powerful mean to reach understanding on questions concerning primes and integers. As is often the case for multiplicative functions and in analysis in general, a major tool for studying characters comes in the form of generating series, the Dirichlet series, which in the special case of a character takes the name of L-function. L-functions provide an invaluable tool to get our hands on prime numbers, as they allow us to transfer sums over integers to products over primes. We will not make much use of L-functions in this work, but we will need this important bridge from integers to primes for a variant of L-functions, so we take a moment to define them here.

Definition 2.1.3. *Let χ be a character modulo q and let s be a complex number with $Re(s) > 1$, then the L-function associated to χ is defined as*

$$L(s,\chi) = \sum_{n \geq 1} \frac{\chi(n)}{n^s}.$$

Now, attached to an L-function is its Euler product, which can be seen as an equivalent definition for the L-function and provides this important bridge between integers and prime numbers.

Proposition 2.1.2. *Let χ be a character modulo q and let s be a complex number with $Re(s) > 1$, then the L-function attached to χ can be written as an Euler product*

$$L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}. \tag{2.1.1}$$

If in a similar manner, we take the product over the primes, but restrict it to the primes up to y, then we recover a sum over y-smooth numbers

$$\prod_{p\leq y} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} = \sum_{\substack{n\geq 1 \\ p|n \implies p\leq y}} \frac{\chi(n)}{n^s}.$$

This is the version that will be useful to us in this thesis, and together with Mertens' theorems, this will enable us to get the proper estimate for Theorem 1.

2.1.2. Gauss sums

Although, we do not make explicit use of Gauss sums in the later work, they do appear and play a role in one of the key ingredient of our proof, so we take the time to define them and cover some simple properties.

Definition 2.1.4. *Let χ be a character modulo q, the* ***Gauss sum*** *$\tau(\chi)$ is defined as*

$$\tau(\chi) = \sum_{n=1}^{q} \chi(n) e\left(\frac{n}{q}\right),$$

where $e(x) = e^{2\pi i x}$.

When studying functions on residue classes such as characters, we often wish to make use of the powerful theory of Fourier analysis. In doing so, it is often necessary to bridge the multiplicative characters χ and the additive characters $e\left(\frac{n}{q}\right)$, and to be able to transfer the problems from the multiplicative side to the additive side and vice-versa. That is the main purpose of the Gauss sum, which can be seen as the inner product between multiplicative and additive characters or as a discrete Fourier transform. We now give a few of its properties that are of interest to us.

A first very useful feature of the Gauss sum is that it allows one to write characters as a linear combination of additive characters $e\left(\frac{an}{q}\right)$.

Theorem 2.1.1. *Let χ be a primitive character modulo q, then*

$$\chi(a) = \frac{1}{\tau(\chi)} \sum_{n=1}^{q} \overline{\chi}(n) e\left(\frac{an}{q}\right), \tag{2.1.2}$$

From this, it is not hard to see that if χ is primitive, then $|\tau(\chi)|^2 = q$, and since we will be using this result, we include the proof here.

Theorem 2.1.2. *Let χ be a primitive character modulo q, then*

$$|\tau(\chi)| = \sqrt{q}.$$

Proof. Multiplying by the conjugate we have

$$\overline{\chi}(a)\chi(a) = \frac{1}{\tau(\chi)} \frac{1}{\tau(\overline{\chi})} \sum_{n_1=1}^{q} \sum_{n_2=1}^{q} \overline{\chi}(n_1)\chi(n_2) e\left(\frac{a(n_1 - n_2)}{q}\right),$$

and summing over all residue classes $a \pmod q$,

$$|\tau(\chi)|^2 \sum_{a=1}^{q} |\chi(a)|^2 = \sum_{n_1=1}^{q} \sum_{n_2=1}^{q} \overline{\chi}(n_1)\chi(n_2) \sum_{a=1}^{q} e\left(\frac{a(n_1 - n_2)}{q}\right).$$

Since the inner sum is zero whenever $n_1 \neq n_2$, then

$$\begin{aligned} \phi(q)|\tau(\chi)|^2 &= q \sum_{n_1=1}^{q} \overline{\chi}(n_1)\chi(n_1) \\ &= q\phi(q). \end{aligned}$$

Hence we deduce that $|\tau(\chi)|^2 = q$ and the result follows. $\square$

2.2. Character sums

Because of their properties, characters provide insights on integers and primes and it is therefore of great interest to understand their behavior. Now, although they respect the same properties, different characters may behave in very different ways locally, and as is the case for general multiplicative functions, a better approach to understanding characters may be to study their mean value. This leads us to investigate the behavior of character sums of the form

$$\sum_{n \leq x} \chi(n),$$

for a positive real number x. In particular, obtaining bounds on such a sum provides valuable information on how the underlying structure of characters affects their behavior. Once again, an efficient way to handle multiplicative functions is often to recourse to Fourier analysis. In the case of character sums, this gives a powerful tool to tackle many questions by putting into play sums containing exponentials, which we can often use to our advantage to exhibit cancellation. We begin this section by proving an important reformulation of character sums via Fourier analysis, before moving to the discussion of some classical and recent results on character sums.

2.2.1. The Pólya-Fourier expansion

In this short section, we prove Pólya's Fourier expansion for character sums. As we will see, this has a particularly great importance for us, as it is the starting point in the proof of Theorem 1.

Theorem 2.2.1. *(Pólya's Fourier expansion for character sums)*
Let χ be a primitive character modulo q, then

$$\sum_{n\leq x}\chi(n)=\frac{\tau(\chi)}{2\pi i}\sum_{\substack{|n|\leq z\\ n\neq 0}}\frac{\overline{\chi}(n)}{n}\left(1-e\left(\frac{-nx}{q}\right)\right)+O\left(1+\frac{q\log z}{z}\right).$$

Proof. For simplicity, let q be prime and for $\alpha\in[0,1]$, define

$$f_\chi(\alpha)=\sum_{n\leq \alpha q}\chi(n),$$

which we extend to a periodic function of period 1 using the orthogonality of characters. That is, the coefficients of the Fourier expansion of $f_\chi(\alpha)$ are given by

$$\begin{aligned}c_\chi(k)&=\int_0^1 f_\chi(\alpha)e(-\alpha k)d\alpha\\ &=\sum_{n=1}^{q}\chi(n)\int_{\frac{n}{q}}^{1}e(-\alpha k)d\alpha.\end{aligned}$$

Now, when $k=0$, then we are just integrating 1, so that

$$c_\chi(0)=\sum_{n=1}^{q}\chi(n)\left(1-\frac{n}{q}\right),$$

and as the orthogonality of characters gives

$$\sum_{n=1}^{q}\chi(n)=0,$$

then this gives

$$c_\chi(0) = \frac{-1}{q}\sum_{n=1}^{q} n\chi(n).$$

When $k \neq 0$, then computing the integral and using orthogonality of characters again, we get

$$\begin{aligned}
c_\chi(k) &= \sum_{n=1}^{q}\chi(n)\left(\frac{e\left(\frac{-kn}{q}\right)-1}{2\pi i k}\right) \\
&= \frac{1}{2\pi i k}\sum_{n=1}^{q}\chi(n)e\left(\frac{-kn}{q}\right) \\
&= \frac{1}{2\pi i}\frac{\overline{\chi}(k)\tau(\chi)}{k},
\end{aligned}$$

by (2.1.2).

Hence, putting this together, we obtain the Fourier expansion for $f_\chi(\alpha)$

$$f_\chi(\alpha) = \frac{-1}{q}\sum_{n=1}^{q} n\chi(n) + \frac{\tau(\chi)}{2\pi i}\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}}\frac{\overline{\chi}(-k)}{k}e(\alpha k),$$

for every point of continuity. Now, observe that for any real numbers $0 < a < b < 1$, then

$$\begin{aligned}
|f_\chi(b) - f_\chi(a)| &= \left|\sum_{aq<n\leq bq}\chi(n)\right| \\
&\leq \sum_{\substack{aq<n\leq bq\\ (n,q)=1}} 1 \\
&= \frac{\phi(q)}{q}(b-a)q + O(2^{\omega(q)}) \\
&\ll \phi(q)(b-a) + 2^{\omega(q)}.
\end{aligned}$$

This means in particular that $f_\chi(\alpha)$ has bounded variation $VAR_{[a,b]} \ll (b-a)\phi(q) + 2^{\omega(q)}$ and so by Theorem 1.1.4,

$$\left|f_\chi(\alpha) - \sum_{|n|\leq K}\hat{f}_\chi(n)e(\alpha n)\right| \leq \int_0^1 \min\left\{\frac{1}{2}, \frac{1}{(2K+1)\pi\sin(\pi x)}\right\} d|f_\chi(\alpha+x)|.$$

Because f is periodic, we have

$$
\begin{aligned}
f_\chi(\alpha) - \sum_{|n|\le K} \hat{f}_\chi(n) e(\alpha n) &= \sum_{k\in\mathbb{Z}} c_\chi(k) e(\alpha k) - \sum_{|n|\le K} e(\alpha n) \sum_{k\in\mathbb{Z}} c_\chi(k)\delta(n-k) \\
&= \sum_{k\in\mathbb{Z}} c_\chi(k) \left(e(\alpha k) - \sum_{|n|\le K} e(\alpha n)\delta(n-k) \right) \\
&= \sum_{|k|>K} c_\chi(k) e(\alpha k),
\end{aligned}
$$

from which we get that

$$
\begin{aligned}
\left| \tau(\chi) \sum_{|k|>K} \frac{\overline{\chi}(-k)}{k} e(\alpha k) \right| &\le \int_0^1 \min\left\{ \frac{1}{2}, \frac{1}{(2K+1)\pi \sin(\pi x)} \right\} d|f_\chi(\alpha+x)| \\
&\le \int_0^{\frac{1}{K}} \frac{1}{2} d|f_\chi(\alpha+x)| + \int_{\frac{1}{K}}^{\frac{K-1}{K}} \frac{1}{(2K+1)\pi \sin(\pi x)} d|f_\chi(\alpha+x)| + \int_{\frac{K-1}{K}}^{1} \frac{1}{2} d|f_\chi(\alpha+x)|
\end{aligned}
$$

By symmetry of the sine function, then

$$
\int_{\frac{1}{K}}^{\frac{K-1}{K}} \frac{1}{\sin(\pi x)} dx = 2 \int_{\frac{1}{K}}^{\frac{1}{2}} \frac{1}{\sin(\pi x)} dx,
$$

so that, splitting the interval of integration, and using our bound for the variation of $f_\chi(\alpha)$, we get

$$
\left| \tau(\chi) \sum_{|k|>K} \frac{\overline{\chi}(-k)}{k} e(\alpha k) \right| \ll \frac{\phi(q)}{K} + \sum_{m=1}^{\frac{K}{2}-1} \int_{\frac{m}{K}}^{\frac{m+1}{K}} \phi(q) \frac{1}{K \sin(\pi x)} dx.
$$

Now, using the fact that $\sin(\pi x) > 2x$ for $x \in \left(0, \frac{1}{2}\right)$, we have

$$\begin{aligned}\left|\tau(\chi)\sum_{|k|>K}\frac{\overline{\chi}(-k)}{k}e(\alpha k)\right| &\ll \frac{\phi(q)}{K}+\sum_{m=1}^{\frac{K}{2}-1}\int_{\frac{m}{K}}^{\frac{m+1}{K}}\phi(q)\frac{1}{Kx}dx\\ &\ll \frac{\phi(q)}{K}+\frac{\phi(q)}{K}\sum_{m=1}^{\frac{K}{2}-1}\left(\log\left(\frac{m+1}{K}\right)-\log\left(\frac{m}{K}\right)\right)\\ &\ll \frac{\phi(q)}{K}+\frac{\phi(q)}{K}\sum_{m=1}^{\frac{K}{2}-1}\left(\log\left(\frac{m+1}{m}\right)\right)\\ &\ll \frac{\phi(q)\log K}{K}.\end{aligned}$$

Hence, we have that

$$f_\chi(\alpha)=\frac{-1}{q}\sum_{n=1}^{q}n\chi(n)+\frac{\tau(\chi)}{2\pi i}\sum_{\substack{|k|\leq z\\ k\neq 0}}\frac{\overline{\chi}(-k)}{k}e(\alpha k)+O\left(1+\frac{q\log z}{z}\right).$$

Finally, differencing we get

$$f_\chi\left(\frac{x}{q}\right)-f_\chi(0)=\frac{\tau(\chi)}{2\pi i}\sum_{\substack{|k|\leq z\\ k\neq 0}}\frac{\overline{\chi}(-k)}{k}\left(e\left(\frac{kx}{q}\right)-1\right)+O\left(1+\frac{q\log z}{z}\right),$$

or in other words

$$\sum_{n\leq x}\chi(n)=\frac{\tau(\chi)}{2\pi i}\sum_{\substack{|k|\leq z\\ k\neq 0}}\frac{\overline{\chi}(k)}{k}\left(1-e\left(\frac{-kx}{q}\right)\right)+O\left(1+\frac{q\log z}{z}\right),$$

as claimed. $\square$

Because characters come up in various settings involving integers, character sums have been extensively studied and are still a very active area of research nowadays. In the following two sections, we first discuss a few classical results that are still at the heart of the subject, and provide a short survey of the motivating papers that have inspired this thesis.

2.2.2. Classical bounds

While studying mean values of arithmetic functions, a natural question arising is concerned with the order of magnitude. More precisely, one wish to find upper bounds when summing the function over integers.

As a consequence of the orthogonality of characters, an easy first estimate for character sums is

$$\sum_{M+1}^{M+N} \chi(n) \leq \min\{N, q\},$$

for any positive M and N.

With a little more work, one can do significantly better. In 1918, Pólya and Vinogradov, independently showed the following.

Theorem 2.2.2 (Pólya-Vinogradov inequality)**.** *Let χ be a non-principal character modulo q, then for any M and $N > 0$*

$$\sum_{n=M+1}^{M+N} \chi(n) \ll \sqrt{q} \log q$$

A hundred years later, this is still the best general unconditional result we have on character sums. It took nearly five decades for a significant improvement to the Pólya-Vinogradov inequality, which was achieved by Burgess in 1962. In a fairly technical paper, he showed what is still today's state of the art.

Theorem 2.2.3 (Burgess bound for characters)**.** *Let q be an odd prime and let χ be a non-principal character modulo q. Further let r be any positive integer, then for $N \geq q^{\frac{1}{4}+\epsilon}$*

$$\sum_{n=M+1}^{M+N} \chi(n) \ll r N^{1-\frac{1}{r}} q^{\frac{r+1}{4r^2}} (\log q)^{\alpha_r},$$

where $\alpha_r = 1$ if $r = 1$ or 2 and $\alpha_r = \frac{1}{2r}$ otherwise.

Note that this improves on the Pólya-Vinogradov inequality only when x is below $\sqrt{q}$. No major improvement has been done to the Burgess bound since, but it is believed that one should expect a little better. Namely, given the widely believed Generalized Riemann hypothesis, it seems reasonable to expect that one can improve on the Burgess bound, as Montgomery and Vaughan have shown in 1977 that

Theorem 2.2.4. *Assume GRH, then we have*

$$\sum_{n=M+1}^{M+N} \chi(n) \ll \sqrt{q} \log \log q.$$

Now, although it is possible to do better when we restrict to specific families of characters (see for example [**8**], [**5**]), there is no hope to do better in the general case. Indeed, in view of Paley's result in 1932 [**15**], this is essentially best possible, as he showed

Theorem 2.2.5. *There exists infinitely many integers q and quadratic characters modulo q such that*

$$\max_{x\leq q}\left|\sum_{n\leq x}\chi(n)\right| \geq \left(\frac{e^{\gamma}}{\pi}+o(1)\right)\sqrt{q}\log\log q.$$

It turns out that in general, we can expect that many odd characters do have large character sums. Indeed, in [**8**], Granville and Soudararajan showed that for a large prime q, not only are there a lot of characters $\pmod q$ whose sum gets large, but we can also find them pointing in every direction.

Theorem 2.2.6. *Let q be a large prime and let $\theta \in (-\pi, \pi]$. Then there is an absolute constant C_0 such that for at least $q^{1-\frac{C_0}{(\log\log q)^2}}$ odd characters $\pmod q$ we have*

$$\sum_{n\leq x}\chi(n) = e^{i\theta}\frac{e^{\gamma}}{\pi}\sqrt{q}\log\log q + O\left((\log\log q)^{\frac{1}{2}}\right)$$

for all but $o(q)$ natural numbers $x \leq q$.

Observe that these theorems show that character sums can get large for many characters, but they give little information as to know in which range of summation such large values occur. We expect that these large values should occur especially when x is a constant multiple of q. As we will see, our main theorem shows that some character sums do get large as x gets closer to q and we believe that we should be able to let the range grow up to q, which would confirm that indeed, very large values such as the one in Theorem 2.2.6 happen at constant multiples of q.

In the optic of better understanding how the values of character sums vary for different ranges of summation, in this thesis, we are considering the question in a different angle. Rather than look at bounds on characters sums when the range is varying, we ask about bounds on characters sums in specific ranges when we let the characters vary. The next section will be devoted to recent results in that direction, that is to say, results concerned with sums of the type

$$\Delta(x,q) := \left|\max_{\chi\neq\chi_0}\sum_{n\leq x}\chi(n)\right|,$$

where x will be in a specific range.

2.2.3. Results on large character sums

Through the knowledge we gain from studying character sums emerge some beliefs of what we think should be the true upper bound for a generic character sum. In order to support these beliefs, we would like to show some omega results, which in this context means finding characters for which the lower bound gets essentially as big as the upper bound, thus showing that they are indeed as good as possible. Moreover, it is expected that long sums of multiplicative functions should exhibit squareroot cancellation, in which case the upper bounds at hand, even under GRH, are good only when x is of size $q^{1-\epsilon}$. Hence, it is of interest to study the extremal values character sums can attain when x is in specified ranges. In that perspective, our quest is about understanding the maximal values that can be achieved by character sums and we do this by considering given ranges over which we take the sum. That is, we wish to study how large

$$\Delta(x,q) = \max_{\chi\neq\chi_0}\left|\sum_{n\leq x}\chi(n)\right| \tag{2.2.1}$$

can get when χ runs through characters modulo q and x is in a given range. In this section, we survey the initial paper that led to this problem and the progress that has been done on it in the recent years.

2.2.3.1. *Large character sums: the genesis of* $\Delta(x,q)$

In their paper "Large character sum" [**7**], published in 1999, Granville and Soundararajan initiated the investigation of the maximal value a character sum could possibly attain. In that paper, they use a variety of techniques allowing them to prove bounds for most ranges for x up to q. As this paper is the motivation behind the work done in this thesis, we now go through the different ranges for x and give a brief exposition on the results they obtain.

Starting with short sums when x go up to roughly $(\log q)^B$, they prove

Theorem 2.2.7. *Let q be a large integer with no prime factors below $\log q$, and suppose $\log x \leq \frac{(\log\log q)^2}{\log\log\log q}$. For all $|\theta|\leq\pi$, there are at least $q^{1-\frac{2}{\log x}}$ characters $\pmod q$ such that*

$$\sum_{n\leq x}\chi(n) = xe^{i\theta}\rho\left(\frac{\log x}{\log\log q}\right)\left(1+O\left(\frac{1}{\log x}+\frac{\log x(\log\log\log q)^2}{(\log\log)^2}\right)\right).$$

Observe that in particular this means that under these conditions

$$\max_{\chi\neq\chi_0}\left|\sum_{n\leq x}\chi(n)\right| \gg x\rho\left(\frac{\log x}{\log\log q}\right),$$

and notice the appearance of the Dickman-De Buijn ρ-function, as is the case in our result.

For the remaining moduli q, they show that

Theorem 2.2.8. *If $x = (10 \log q)^B$, for some $B \geq 1$, then*

$$\max_{\chi \neq \chi_0} \left| \sum_{n \leq x} \chi(n) \right| \gg \frac{x^{\frac{1}{2} + \frac{\lfloor B \rfloor}{2B}}}{(4 \log x)^{\lfloor B \rfloor}},$$

from which they are able to deduce

Corollary 2.2.1. *Fix σ in the range $\frac{1}{2} \leq \sigma < 1$. If $(\log q)^{\frac{1}{1-\sigma}} \leq x \leq \exp(((\log q)^{1-\sigma+o(1)})$ then*

$$\max_{\chi \neq \chi_0} \left| \sum_{n \leq x} \chi(n) \right| \gg x^{\sigma}.$$

Next, when x is in the the range $\log \log x = (\frac{1}{2} + o(1)) \log \log q$, they prove

Theorem 2.2.9. *Suppose that $\log x = \tau \sqrt{\log \log \log q}$ with $\tau = (\log \log q)^{O(1)}$ and let $\eta = \tau + 1/\tau$. Then for q large enough, there exists a constant $c > 0$ such that*

$$\max_{\chi \neq \chi_0} \left| \sum_{n \leq x} \chi(n) \right| \gg \sqrt{x} \exp\left(c \frac{1 + \log(\eta \tau)}{\eta} \sqrt{\frac{\log q}{\log \log q}} \right).$$

They go on investigating the range when $\frac{\log x}{\sqrt{\log q \log \log q}}$ is large but x is smaller than any power of q, and they get

Theorem 2.2.10. *Suppose both $\frac{\log q}{\log x}$ and $\frac{\log x}{\log \log \log q} \to \infty$, then*

$$\max_{\chi \neq \chi_0} \left| \sum_{n \leq x} \chi(n) \right| \gg \sqrt{x} \left(\frac{\log x}{\sqrt{\log q \log \log q}} \right)^{(1+o(1)) \frac{\log q}{\log x}}.$$

Finally, when x is as big as a power of q, Granville and Soundararajan show

Theorem 2.2.11. *Let $k \geq 2$ be an integer and suppose $\exp\left(\frac{\log q}{\log \log q}\right) \leq x \leq q^{\frac{1}{k}}$, then*

$$\max_{\chi \neq \chi_0} \left| \sum_{n \leq x} \chi(n) \right| \gg_k \sqrt{x} (\log q)^{\frac{(k-1)^2}{2k} + o(1)}.$$

Although the theorems for the various ranges are proven in different ways, there are some key ideas pertaining to the whole paper that form the basis for the proofs. The central idea resides in the computation of the $2k$-th moments of the character sums. In order to do so, the authors use a clever transfer from moments of character sums to expectations of independent random variables. This allows them to use tools from probability theory to get a hold on the moments. This is a crucial idea in the paper which is used repeatedly, and together with the restriction of the character sums over integers up to x to character sums over y-smooth numbers where y is roughly $\log q$, they are able to get a good understanding of $\Delta(x, q)$.

Now, the ranges for which they get non-trivial results go up to $\sqrt{q}$, as past this, the bound follows from simple mean square and gives $\Delta(x, q) \geq \sqrt{x} \log q^{o(1)}$. Beyond $\sqrt{q}$, they observe that using Pólya's Fourier expansion one should be able to use the same line of ideas to get non-trivial bounds for 2.2.1 when x ranges between $\sqrt{q}$ and q. However, they are unable to use these ideas to prove lower bounds that hold for every x in the range, but rather obtain results for some $t \leq x$. Our theorem makes use of this idea in a fruitful way.

In the optic of comparing the bound in Theorem 2.2.11 with the one we obtain in this thesis, we note that given x of the type $x = \frac{q}{(\log q)^B}$, mean square estimates give

$$\max_{\chi \neq \chi_0} \left| \sum_{n \leq \frac{q}{(\log q)^B}} \chi(n) \right| \gg \frac{\sqrt{q}}{(\log q)^{\frac{B}{2} - o(1)}}. \tag{2.2.2}$$

2.2.3.2. *Subsequent progress on* $\Delta(x, q)$

Granville and Soundararajan's paper [**7**] provided an extensive study of $\Delta(x, q)$ and it took a few years for the question to be raised again. Using a new technique called the resonance method introduced by Soundararajan in 2007, Bob Hough reattacked the problem in his 2011 paper [**10**], improving on the bounds for all the ranges covered in [**7**], when x is greater than a small power of $\log q$. Moreover, using the resonance method, he was able to push the range to obtain a non-trivial lower bound for $\Delta(x, q)$ past $\sqrt{q}$ all the way to $x \leq q^{1-\epsilon}$. We will not cover all the ranges here, but we do state the result Hough obtained for this extra range. Here is what he proved.

Theorem 2.2.12. *Let q be a large integer and $4\sqrt{\log q \log\log q}(\log\log\log q) \leq \log x$ and $x = q^\theta$, for $\theta < 1 - \epsilon$. Then*

$$\max_{\chi \neq \chi_0} \left| \sum_{n \leq x} \chi(n) \right| \gg \sqrt{x} \exp\left((1 + o(1)) \sqrt{\frac{(1-\theta)\log q}{\log\log q}} \right).$$

With the same technique, Munsch in [**14**] (2018) succeeded in proving new lower bounds for $\Delta(x, q)$, when $\log x = o(\log q)$, in particular, his bound improves known bounds for

$\log x \leq (\log q)^{\frac{1}{2}}$. He proved the following

Theorem 2.2.13. *Let q be a large prime and suppose $\log q < x \leq \exp(\sqrt{\log q})$. Let $\nu = \max\{\log\log x - \log\log\log q, \log\log\log q\}$, then*

$$\max_{\chi \neq \chi_0} \left| \sum_{n \leq x} \chi(n) \right| \gg \psi\left(x, \left(\frac{1}{4} + o(1)\right) \frac{\log q \log\log q}{\nu}\right).$$

In passing, we note that in a very recent paper (2019), De la Bretèche, Munsch and Tenenbaum obtained an interesting result on $\Delta(x, q)$ that stands out by its uniformity on a very wide range. In their paper [**4**], the authors obtain lower bounds for the low moments of character sums, and in particular, they get a lower bound for the first moment of character sums, that is to say the average, which in turn provides a lower bound for the size of $\Delta(x, q)$. More precisely, they show

Theorem 2.2.14. *Let q be a large prime, let $1 \leq x < \frac{q}{2}$ and let $\nu = \min\{x, \frac{q}{x}\}$. Then for $c = 0.04305$,*

$$\frac{1}{q-2} \sum_{\chi \neq \chi_0} \left| \sum_{n \leq x} \chi(n) \right| \gg \frac{\sqrt{x}}{(\log \nu)^c (\log\log \nu)^3}.$$

This implies in particular that

$$\max_{\chi \neq \chi_0} \left| \sum_{n \leq x} \chi(n) \right| \gg \frac{\sqrt{x}}{(\log \nu)^c (\log\log \nu)^3}.$$

Although their estimate is not quite as good as the one from Granville and Soundararajan [**7**], we highlight the fact that the result in Theorem 2.2.14 is still remarkable for its uniformity on such a range.

Now, one of the strength of Granville and Soundararajan's paper [**7**] is that it non-only gives a lower bound for $\Delta(x, q)$, but it actually gives in many cases a lower bound for the proportion of the characters for which the bounds hold. This leads to the question of understanding how characters are distributed according to the maximal size of their character sums. That is, one would like to understand the distribution of

$$M(\chi) = \max_{1 \leq x \leq q} \left| \sum_{n \leq x} \chi(n) \right|,$$

for the different characters modulo q.

The problem actually goes back to Montgomery and Vaughan in their 1979 paper [**12**]. Indeed, if we let $m(\chi) = \frac{M(\chi)}{e^\gamma \sqrt{q}/\pi}$ and we define the following distribution function

$$\Phi_q(\tau) = \frac{1}{\phi(q)} \# \left\{ \chi \pmod q : m(\chi) > \tau \right\},$$

they showed that for any $\tau > 1$ and any fixed constant $C \geq 1$, then

$$\Phi_q(\tau) \ll_C \tau^{-C}.$$

With the study of maximum of character sums in recent years, the problem came up again and in 2011, Bober and Goldmakher improved on Montgomery and Vaughan's result by showing that, for fixed τ and primes q going to infinity, we have

$$\exp\left\{ -\frac{e^{\tau+A}}{\tau} \left(1 + O\left(\frac{1}{\sqrt{\tau}} \right) \right) \right\} \leq \Phi_q(\tau) \leq \exp\left\{ -Be^{\frac{\sqrt{\tau}}{(\log \tau)^{1/4}}} \right\}, \tag{2.2.3}$$

where $A \approx 0.088546$ is a fixed constant and B is a positive constant.

This was subsequently improved in 2014 by Bober. Goldmakher, Granville and Koukoulopoulos in [**2**], in which they proved

Theorem 2.2.15. *Let $\eta = e^{-\gamma} \log 2$. If q is a prime and $1 \leq \tau \leq \log\log q - M$ for some $M \geq 4$, then*

$$\exp\left\{ -\frac{e^{\tau+A-\eta}}{\tau} \left(1 + O\left(\epsilon_1 \right) \right) \right\} \leq \Phi_q(\tau) \leq \exp\left\{ -\frac{e^{\tau-2-\eta}}{\tau} \left(1 + O\left(\epsilon_2 \right) \right) \right\},$$

where

$$\epsilon_1 = \tfrac{(\log \tau)^2}{\sqrt{\tau}} + e^{-M/2} \quad \text{and} \quad \epsilon_2 = \tfrac{\log \tau}{\tau}.$$

As is the case in our theorem, they observe that the large sums arise from odd characters, and thus they go on investigating the distribution of maximums of both even and odd character sums separately. Moreover, they show that the distribution function $\Phi_q(\tau)$ tends to a universal distribution $\phi(\tau)$ as q goes to infinity. Their paper gives a wonderful insight on the size of character sums, showing among other things that $m(\chi)$ rarely gets large. The proof of our theorem borrows some ideas and results from the proof of Theorem 2.2.15, in which the core ideas involve using Pólya's Fourier expansion, and estimating it by showing that for most characters, the main contribution comes from smooth numbers. The difficulty of the proof actually lies in showing that for most characters, the character sum restricted to rough numbers (by opposition to smooth numbers) is small. To do so, the authors bring back ingredients from [**7**], namely by computing high moments of the restricted character sums and using different tools, including expectations, to bound the moments. Although we will not use these last tools, as we will see, our proof does use a truncated Pólya's Fourier expansion restricted on smooth numbers, as they do.

Chapter 3

A first result about lattices

One of the main challenges in the proof of Theorem 1 arises from finding an odd character which takes values close to one on all primes up to some point T. In order to handle this obstacle, we make a slight digression in the world of lattices, which will result in theorems on lattices that are interesting on their own. As this part of the proof of Theorem 1 differs greatly from the rest and bares an independent interest, we treat it separately in the present chapter.

In this chapter, we let M be a positive integer and we let $\mathbf{u} = \frac{1}{M}(u_1, u_2, \cdots, u_k)$ be a lattice vector in $(\mathbb{R}/\mathbb{Z})^k$ of order M, that is to say that M is the smallest integer such that $M\mathbf{u} \equiv \mathbf{0} \pmod 1$.

3.1. The easier case: Vector multipliers $\ell \equiv 0 \pmod n$

We first show without too much effort that we can find many multiples $\mathbf{x}_\ell \equiv \ell\mathbf{u} \pmod 1$ with only small components. So define

$$C_{n^+}(\eta, k) = \{0 \leq \ell \leq M-1 \text{ , } \ell \equiv 0 \pmod n : |x_{\ell,j}| \leq \eta \ \forall 1 \leq j \leq k\}, \tag{3.1.1}$$

and

$$C_{n^-}(\eta, k) = \{0 \leq \ell \leq M-1 \text{ , } \ell \not\equiv 0 \pmod n : |x_{\ell,j}| \leq \eta \ \forall 1 \leq j \leq k\}, \tag{3.1.2}$$

where $x_{\ell,j}$ is the j^{th} component of the vector $\mathbf{x}_\ell \equiv l\mathbf{u} \pmod 1$.

Recall Proposition 1:

Proposition 1 *Let N be a positive real number and let $\mathbf{u} \in (\mathbb{R}/\mathbb{Z})^k$ be a k-dimensional lattice vector of order M. Then for any fixed integer $n < \frac{M}{N^k}$,*

$$\#C_{n^+}\left(\frac{1}{N}, k\right) \geq \frac{M}{nN^k}.$$

As an immediate corollary, we get

Corollary 3.1.1. *Let* $\mathbf{u} \in (\mathbb{R}/\mathbb{Z})^k$ *be a lattice vector of order* M*, then*

$$\#C_{2^+}\left(\frac{1}{N}, k\right) \geq \frac{M}{2N^k}.$$

Proof of Proposition 1. Let $\mathbf{x}_\ell \equiv \ell \mathbf{u} \pmod 1$, where $n < \frac{M}{N^k}$ is fixed and the multipliers $0 \leq \ell \leq M-1$ satisfies $\ell \equiv 0 \mod n$. We split $(\mathbf{R}/\mathbf{Z})^k$ into N^k equal hypercubes, each side of which has length $1/N$. Notice that for each integer $0 \leq \ell \leq M-1$, with $l \equiv 0 \pmod n$, the vector $\mathbf{x}_l$ must belong to one of the cubes, and therefore, by the pigeonhole principle, we must have an hypercube C which contains at least $\frac{M}{nN^k}$ vectors.

Now, fix $\mathbf{x}_r \in C$ where $r > s$ for all other vectors $\mathbf{x}_s \in C$. By the construction of the cubes, for any other vector $\mathbf{x_s}$ in C we must have $|x_{r,j} - x_{s,j}| \leq \frac{1}{N}$, for $j \leq k$. Now let $\ell = r - s$ and observe that $r - s \equiv 0 \mod n$ and thus the vector $\mathbf{x}_\ell = \mathbf{x_r} - \mathbf{x_s} \equiv (r-s)\mathbf{u} \pmod 1$ has multiplier $\ell \equiv 0 \pmod n$, with each component of size at most $1/N$. As there are $\frac{M}{nN^k}$ such vectors $\mathbf{x_s} \in C$, including $\mathbf{x}_r$, we deduce that there are at least $\frac{M}{nN^k}$ integers $0 \leq \ell \leq M-1$, with $\ell \equiv 0 \mod n$, such that $\mathbf{x}_\ell$ has components $|x_{\ell,j}| \leq \frac{1}{N}$ for all $j \leq k$ and the result follows. $\square$

Corollary 3.1.1 is important for us, as it will allow us to show the existence of many even characters with small argument. However, even more important to us is to show that there are a lot of odd characters with small arguments.

Now, the argument holds when the multiplier for $\mathbf{u}$ is of the form $\ell \equiv 0 \pmod n$, since taking the difference of any two vectors in the hypercube C will produce a vector whose multiplier will also satisfy $r - s \equiv 0 \pmod n$. However it does not hold for the complementary set of integers $\ell \not\equiv 0 \mod n$, as in that case, we could have vectors $\ell_1 \equiv a \pmod n$ and $\ell_2 \equiv a \pmod n$ in C, so that the difference would produce a small vector with multiplier $\ell_1 - \ell_2 \equiv 0 \pmod n$, which is not what we are looking for. As our argument does not allow us to detect when this happens, it cannot be used to count small vectors with multipliers satisfying the condition $\ell \not\equiv 0 \pmod n$. However, the next lemma shows that if we can find just one vector with multiplier $\ell \not\equiv 0 \pmod n$ that is small, then in view of Theorem 1, we can actually find many of them.

Lemma 3.1.1. *Given a lattice vector* $\mathbf{u} \in (\mathbb{R}/\mathbb{Z})^k$ *of order* M*, let* $\mathbf{x}_\ell \equiv \ell \mathbf{u} \pmod 1$*.*

Suppose that $C_{n^-}(\nu, k) \neq \emptyset$*, then*

$$\#C_{n^-}(\nu + \eta, k) \geq \#C_{n^+}(\eta, k),$$

where the sets are defined as in (3.1.1) and (3.1.2).

Proof. Suppose that there exists an integer $0 \le r \le M-1$, with $r \not\equiv 0 \mod n$, such that each component of $\mathbf{x}_r$ satisfies $|x_{r,j}| \le \nu$ for $1 \le j \le k$. For any integer $s \equiv 0 \pmod{n}$ in the same range and such that the vector $\mathbf{x}_s \in C_+(\eta, k)$, then $\ell \equiv r - s \pmod{M}$ satisfies $\ell \not\equiv 0 \pmod{n}$ and the size of the components of the vector $\mathbf{x}_\ell$ is bounded by

$$\begin{aligned} |x_{\ell,j}| &= |x_{r,j} \pm x_{s,j}| \\ &\le |x_{r,j}| + |x_{s,j}| \\ &\le \eta + \nu. \end{aligned}$$

Hence, it follows that $\mathbf{x}_\ell \in C_-(\eta+\nu, k)$. Now it is easy to see that distinct vectors in $C_+(\eta, k)$ will give rise to distinct vectors in $C_-(\eta+\nu, k)$, and therefore it follows that $\#C_-(\eta+\nu, k) \ge \#C_+(\eta, k)$. $\square$

3.2. The harder case: Vector multipliers $\ell \not\equiv 0 \pmod{n}$

As we will see, taking multipliers of the form $\ell \not\equiv 0 \pmod{n}$ for our lattice vector $\mathbf{u}$ is more subtle and the existence of a small vector of that form is not always guaranteed. Yet an interesting phenomenon occurs and we extract a condition for a small vector to exist.

Theorem 3 *Let $N > 0$, k be a large integer and let $\mathbf{u} \in (\mathbb{R}/\mathbb{Z})^k$ be a lattice vector of order M. Given a divisor n of M then either*

(i) *There exists a non-zero vector $\mathbf{r} \in (\mathbb{R}/\mathbb{Z})^k$ such that $|r_j| \le k^4 N \log^2(N)$ for $j \le k$ and $n(\mathbf{r} \cdot \mathbf{u}) \equiv 0 \pmod{1}$,*

or

(ii)
$$\#C_{n^-}\left(\frac{2}{N}, k\right) \ge \frac{M}{nN^k}.$$

Remark. *We stress the fact that in Theorem 3, n is limited to the divisors of M, as opposed to all integers as in Proposition 1.*

Theorem 3 follows directly from Proposition 1, Lemma 3.1.1 and the following key proposition.

Proposition 3.2.1. *Let $N > 0$, k be a large integer and let $\mathbf{u} \in (\mathbb{R}/\mathbb{Z})^k$ be a lattice vector of order M. Given a divisor n of M then either* *(i) There exists a non-zero vector* $\mathbf{r} \in (\mathbb{R}/\mathbb{Z})^k$ *such that* $|r_j| \leq k^4 N \log^2(N)$ *for* $j \leq k$ *and* $n(\mathbf{r} \cdot \mathbf{u}) \equiv 0 \pmod 1$*,*

or

(ii)

$$C_{n^-}\left(\frac{1}{N}, k\right) \neq \emptyset.$$

Proof. In order to prove Proposition 3.2.1, we construct a counting function detecting vectors with small components and we appeal to Fourier analysis to show that the counting function is non-negative when applied to vectors of the form $\mathbf{x}_\ell \equiv \ell\mathbf{u} \pmod 1$, proving the existence of such a short vector.

So, let $\mathbf{u} \in (\mathbb{R}/\mathbb{Z})^k$ be a given lattice vector of order M, that is $\mathbf{u} = \frac{1}{M}(u_1, u_2, \cdots, u_k)$, and let n be any divisor of M, so that $M = nm$. For now, suppose that there is a positive real number L for which there is no vector $\mathbf{r} \in \mathbb{Z}^k$, with $|r_j| < L$ for $j \leq k$ such that $n(\mathbf{r} \cdot \mathbf{u}) \equiv 0 \pmod 1$.

In order to detect the vectors we want, we start by choosing a positive valued Schwartz function with support $\left(-\frac{1}{2}, \frac{1}{2}\right)$. We choose

$$\phi(x) = \begin{cases} c_0 e^{\frac{-1}{1-(2x)^2}} & \text{if } x \in \left[-\frac{1}{2}, \frac{1}{2}\right] \\ 0 & \text{otherwise,} \end{cases}$$

where c_0 is a normalizing constant so that

$$\int_{-\infty}^{\infty} \phi(x)dx = 1,$$

and we scale it so that it has support in $\left[-\frac{1}{2N}, \frac{1}{2N}\right]$ by letting

$$\phi_N(x) = N\phi(Nx).$$

Next, define the k-dimensional bump function

$$\Phi(\mathbf{x}) = \prod_{j \leq k} \phi(x_j),$$

and notice that the function is non-negative and has support inside $\left[-\frac{1}{2N}, \frac{1}{2N}\right]^k$ so that whenever the function is non-zero, then we have a vector whose components all lie within the interval $\left(\frac{-1}{N}, \frac{1}{N}\right)$.

Next, as we wish to catch the vectors with components in that interval modulo 1, we extend our function by shifting by every integer vector

$$F_N(\mathbf{x}) = \sum_{\mathbf{v}\in\mathbb{Z}^k} \Phi_N(\mathbf{v}-\mathbf{x}).$$

With this in hand, we can finally define our counting function which, for a generic set of vectors V, takes the form

$$S(N) = \sum_{\mathbf{x}\in V} F_N(\mathbf{x}).$$

Now, as we are actually interested by the set of vectors $V = \{\mathbf{x} \equiv \ell\mathbf{u} \pmod 1 : 1 \leq \ell \leq M-1\ ,\ \ell \not\equiv 0 \pmod n\}$, we write

$$S(N) = \sum_{a=1}^{n-1} \sum_{\substack{1\leq \ell \leq M-1 \\ \ell\equiv a \pmod n}} F_N(\ell\mathbf{u}).$$

The objective is to show that $S(N) > 0$, from which we will deduce that the detecting function $F_N(\ell\mathbf{u})$ is non-zero for some odd integer ℓ, thus proving the existence of a vector $\mathbf{x} \equiv \ell\mathbf{u} \pmod 1$ with components in $\left(\frac{-1}{N}, \frac{1}{N}\right)$.

By Poisson summation formula, we have that

$$\begin{aligned}
F_N(\mathbf{x}) &= \sum_{\mathbf{r}\in\mathbb{Z}^k} e(\mathbf{x}\cdot\mathbf{r})\hat{\Phi}_N(\mathbf{r}) \\
&= \sum_{\mathbf{r}\in\mathbb{Z}^k} e(\mathbf{x}\cdot\mathbf{r})\frac{1}{N^k}\hat{\Phi}_N\left(\mathbf{r}\right) \\
&= \sum_{\mathbf{r}\in\mathbb{Z}^k} \frac{e(\mathbf{x}\cdot\mathbf{r})}{N^k}\hat{\Phi}_N\left(\mathbf{r}\right),
\end{aligned}$$

therefore substituting, we get

$$\begin{aligned}
S(N) &= \sum_{a=1}^{n-1} \sum_{\substack{1\leq \ell \leq M-1 \\ \ell\equiv a \pmod n}} \sum_{\mathbf{r}\in\mathbb{Z}^k} \frac{e(\mathbf{r}\cdot\ell\mathbf{u})}{N^k}\hat{\Phi}_N\left(\mathbf{r}\right) \\
&= \frac{1}{N^k}\sum_{\mathbf{r}\in\mathbb{Z}^k} \hat{\Phi}_N\left(\mathbf{r}\right) \sum_{a=1}^{n-1} \sum_{0\leq s\leq m-1} e((sn+a)\mathbf{u}\cdot\mathbf{r}).
\end{aligned}$$

Now, the inner sum is

$$e(a\mathbf{u}\cdot\mathbf{r}) \sum_{0\leq s\leq m-1} e(sn\mathbf{u}\cdot\mathbf{r}),$$

and notice that all components of $\mathbf{u}$ are of the form $\frac{u_j}{M}$ for some $1 \leq u_j \leq M-1$, so that we have a complete exponential sum and thus

$$\sum_{0 \leq s \leq m-1} e(sn\mathbf{u} \cdot \mathbf{r}) = \begin{cases} m & \text{if } n(\mathbf{u} \cdot \mathbf{r}) \equiv 0 \pmod 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, we have that

$$S(N) = \frac{m}{N^k} \sum_{\substack{r \in \mathbb{Z}^k \\ n(\mathbf{r} \cdot \mathbf{u}) \equiv 0 \pmod 1}} \hat{\Phi}_N(\mathbf{r}) \sum_{a=1}^{n-1} e(a\mathbf{r} \cdot \mathbf{u}).$$

Observe that

$$\sum_{a=1}^{n-1} e(a\mathbf{r} \cdot \mathbf{u}) = \begin{cases} n-1 & \text{if } \mathbf{r} \cdot u \equiv 0 \pmod 1, \\ -1 & \text{otherwise.} \end{cases}$$

Therefore, it follows that

$$S(N) = \frac{m}{N^k} \left((n-1) \sum_{\substack{\mathbf{r} \in \mathbb{Z}^k \\ \mathbf{r} \cdot \mathbf{u} \equiv 0 \pmod 1}} \hat{\Phi}_N(\mathbf{r}) - \sum_{\substack{\mathbf{r} \in \mathbb{Z}^k \\ n(\mathbf{r} \cdot \mathbf{u}) \equiv 0 \pmod 1 \\ \mathbf{r} \cdot \mathbf{u} \not\equiv 0 \pmod 1}} \hat{\Phi}_N(\mathbf{r}) \right).$$

As we expect when working with Fourier transforms, only the small values of $\mathbf{r}$ have an important contribution, so that we can truncate the sums without to much loss. Indeed,

$$\begin{aligned} \left| \sum_{\substack{\mathbf{r} \in \mathbb{Z}^k \\ \max_{j \leq k} |r_j| > L \\ \mathbf{r} \cdot \mathbf{u} \equiv 0 \pmod 1}} \hat{\Phi}_N(\mathbf{r}) - \sum_{\substack{\mathbf{r} \in \mathbb{Z}^k \\ \max_{j \leq k} |r_j| > L \\ n(\mathbf{r} \cdot \mathbf{u}) \equiv 0 \pmod 1 \\ \mathbf{r} \cdot \mathbf{u} \not\equiv 0 \pmod 1}} \hat{\Phi}_N(\mathbf{r}) \right| &\leq \sum_{j=1}^{k} \sum_{\substack{\mathbf{r} \in \mathbb{Z}^k \\ |r_j| > L}} |\hat{\Phi}_N(\mathbf{r})| \\ &\leq k \sum_{\substack{r \in \mathbb{Z} \\ |r| > L}} |\hat{\phi}_N(r)| \left(\sum_{r \in \mathbb{Z}} |\hat{\phi}_N(r)| \right)^{k-1} \\ &\leq k \int_{|t| > L} \left| \hat{\phi}\left(\frac{t}{N}\right) \right| dt \left(\int_{-\infty}^{\infty} \left| \hat{\phi}\left(\frac{t}{N}\right) \right| dt \right)^{k-1} \end{aligned}$$

Observe that as $\phi(x)$ is a Schwartz function, so is $\hat{\phi}(y)$ which means that it is in L^1 and thus, with the appropriate change of variable,

$$\int_{-\infty}^{\infty} \left|\hat{\phi}\left(\frac{t}{N}\right)\right| dt = cN,$$

for some absolute constant c.

Moreover, knowing that $|\hat{\phi}(y)| \leq \frac{2}{N}\frac{e^{-\sqrt{y}}}{y^{3/4}}$ (see [**11**]) for large enough positive $y \in \mathbb{R}$, we have

$$\begin{aligned}
\int_{|t|>L} \left|\hat{\phi}\left(\frac{t}{N}\right)\right| dt &\leq \int_{|t|>L} \frac{e^{-\sqrt{\frac{|t|}{N}}}}{\left(\frac{|t|}{N}\right)^{\frac{3}{4}}} dt \\
&\leq 2\int_{t>L} \frac{e^{-\sqrt{\frac{t}{N}}}}{\sqrt{\frac{t}{N}}} dt \\
&= 2\int_{v>\sqrt{\frac{L}{N}}} e^{-v} dv \\
&= 2e^{-\sqrt{\frac{L}{N}}}.
\end{aligned}$$

Hence, putting this together we have

$$k\int_{|t|>L} \left|\hat{\phi}\left(\frac{t}{N}\right)\right| dt \left(\int_{-\infty}^{\infty} \left|\hat{\phi}\left(\frac{t}{N}\right)\right| dt\right)^{k-1} \leq 2k(Nc)^{k-1} e^{-\sqrt{\frac{L}{N}}},$$

and choosing $L = k^4 N \log^2(N)$, we get that

$$\left| \sum_{\substack{\mathbf{r}\in\mathbb{Z}^k \\ \max_{j\leq k}|r_j|>L \\ \mathbf{r}\cdot\mathbf{u}\equiv 0 \pmod 1}} \hat{\Phi}_N(\mathbf{r}) - \sum_{\substack{\mathbf{r}\in\mathbb{Z}^k \\ \max_{j\leq k}|r_j|>L \\ n(\mathbf{r}\cdot\mathbf{u})\equiv 0 \pmod 1 \\ \mathbf{r}\cdot\mathbf{u}\not\equiv 0 \pmod 1}} \hat{\Phi}_N(\mathbf{r}) \right| \ll \frac{kc^{k-1}}{N^{k^2-k}}.$$

This means that we can truncate the sum to get

$$S(N) = \frac{m}{N^k}\left((n-1) \sum_{\substack{\mathbf{r}\in\mathbb{Z}^k \\ |r_j|<L \\ \mathbf{r}\cdot\mathbf{u}\equiv 0 \pmod 1}} \hat{\Phi}_N(\mathbf{r}) - \sum_{\substack{\mathbf{r}\in\mathbb{Z}^k \\ |r_j|<L \\ n(\mathbf{r}\cdot\mathbf{u})\equiv 0 \pmod 1 \\ \mathbf{r}\cdot\mathbf{u}\not\equiv 0 \pmod 1}} \hat{\Phi}_N(\mathbf{r}) + o(1) \right).$$

Now, by hypothesis, there are no non-zero vectors $|r_j| \leq L$ satisfying $n(\mathbf{r}\cdot\mathbf{u}) \equiv 0 \pmod 1$, hence it follows that most of the sums are empty and the only term left is $\hat{\Phi}_N(0)$. It follows that

$$S(N) = \frac{m}{n^k}\left((n-1)\hat{\Phi}_N(0) + o(1)\right).$$

Computing

$$\begin{aligned}\hat{\Phi}_N(0) &= \left(\int_{-\infty}^{\infty} \phi_N(t)dt\right)^k \\ &= 1,\end{aligned}$$

we deduce that

$$S(N) > 0,$$

which shows that there is an integer $\ell \not\equiv 0 \pmod n$ with $1 \leq \ell \leq M-1$, such that $F_N(\ell\mathbf{u}) > 0$. As a consequence, we conclude that there is such an integer ℓ such that if $\mathbf{x} \equiv \ell\mathbf{u} \pmod 1$, then $|x_j| < \frac{1}{N}$ for all $1 \leq j \leq k$. $\square$

As it will play a role in the proof of Theorem 1, we highlight the case $n = 2$.

Corollary 3.2.1. *Let $\mathbf{u} \in (\mathbb{R}/\mathbb{Z})^k$ be a vector of order $2m$, and suppose that there is no vector $\mathbf{r} \in \mathbb{R}^k$ with $|r_j| < k^4 N(\log N)^2$ for all $j \leq k$, such that $2(\mathbf{r}\cdot\mathbf{u}) \equiv 0 \mod 1$, then*

$$C_{2^-}\left(\frac{2}{N}, k\right) \geq \frac{M}{2N^k}.$$

.

Next, we show that a similar result holds in the opposite situation. Indeed, the next theorem shows that if $\mathbf{r}\cdot\mathbf{u} \equiv \frac{t}{n} \pmod 1$, then for any integer $\ell \not\equiv 0 \pmod n$, the vectors $\mathbf{x} \equiv \ell\mathbf{u} \pmod 1$ and $\mathbf{r}$ cannot be small at the same time.

Theorem 4 *Let $\mathbf{u} = \frac{1}{M}(u_1, \cdots, u_k) \in (\mathbb{R}/\mathbb{Z}^k)$ be a k-dimensional lattice vector of order M and let n be any integer. Suppose that there exists $\mathbf{r} \in \mathbb{Z}^k$ such that $\mathbf{r}\cdot\mathbf{u} \equiv \frac{t}{n} \pmod 1$, where $(t, n) = 1$. Then for any integer $1 \leq \ell \leq M-1$ such that $\ell \not\equiv 0 \pmod n$, the vector $\mathbf{x} = \ell\mathbf{u} \pmod 1 \in (\mathbb{R}/\mathbb{Z})^k$ satisfies*

$$|\mathbf{r}\cdot\mathbf{x}| \geq \frac{1}{n}. \tag{3.2.1}$$

In particular, if $\mathbf{r} \cdot \mathbf{u} \equiv 0 \pmod 1$ *than*

$$|\mathbf{x}| \geq \frac{1}{|\mathbf{r}|}.$$

PROOF. Suppose that $\mathbf{r} \cdot \mathbf{u} \equiv \frac{t}{n} \pmod 1$, write $\ell = sn + a$ and consider

$$\begin{aligned}
\mathbf{r} \cdot \mathbf{x} &= \ell(\mathbf{r} \cdot \mathbf{u}) \\
&= (sn + a)(\mathbf{r} \cdot \mathbf{u}) \\
&\equiv (sn + a)\frac{t}{n} \pmod 1 \\
&\equiv \frac{at}{n} \pmod 1.
\end{aligned}$$

Since $(t, n) = 1$ and $a < n$, then $at \not\equiv 0 \pmod n$, thus it follows that

$$|\mathbf{r} \cdot \mathbf{x} \pmod 1| \geq \frac{1}{n},$$

which proves the first part of the theorem.

The second part follows directly from the observation that

$$\begin{aligned}
|\mathbf{r}||\mathbf{x}| \cos\theta &= \mathbf{r} \cdot \mathbf{u} \\
&\geq \frac{1}{n},
\end{aligned}$$

so that

$$|\mathbf{x}| \geq \frac{1}{|\mathbf{r}|n}.$$

□

In the next chapter, we apply these results to the main object of study in this thesis and show how they help us get information on characters.

Chapter 4

When χ pretends to be 1

A common strategy when looking for lower bounds for a family of functions is to find extremal cases and use the particular instances to derive estimates. In the case of character sums, this means working with characters that pretend to be 1, that is to say characters taking values close to 1 on all the small primes. It is believed that there are characters taking value 1 for all the primes $p \ll (\log q)^{1-\epsilon}$, but showing this is out of reach, so we resort to a softer condition. Instead, we will consider that a character pretends to be 1 if it is in the sets

$$A_{\pm}(N,T) = \left\{\chi \pmod q : \chi(-1) = \pm 1 \ , \ \max_{p\leq T}|\chi(p)-1| \ll \frac{1}{N}\right\}, \tag{4.0.1}$$

where $T \geq 2$ and $N = N(T) \to \infty$ as $T \to \infty$.

In this chapter, we investigate some repercussions of supposing that a bound such as the one in $A_{\pm}(T,N)$ holds for a character and then proceed to confirm that the sets $A_{\pm}(T,N)$ do indeed contain many characters.

4.1. What if χ pretends to be 1?

Proposition 4.1.1. *Suppose that* $\chi \in A_{\pm}(T,N)$ *, and let* $\log\log y = \left(1 + O\left(\frac{1}{N}\right)\right)\log\log T$*, then*

$$\sum_{p\leq y}\frac{\chi(p)-1}{p} \ll \frac{\log\log T}{N}.$$

Proof. Using the bound from (4.0.1), $\max_{p \le T} |\chi(p) - 1| \ll \frac{1}{N}$, together with Theorem 1.2.3, we have

$$\begin{aligned}\sum_{p \le y} \frac{\chi(p) - 1}{p} &= \sum_{p \le T} \frac{\chi(p) - 1}{p} + O\left(\log\left(\frac{\log y}{\log T}\right)\right) \\ &\le \max_{p \le T} |\chi(p) - 1| \sum_{p \le T} \frac{1}{p} + O\left(\frac{\log\log T}{N}\right) \\ &\ll \frac{\log\log T}{N}.\end{aligned}$$

□

From now on, define

$$h(T) = \frac{\log\log T}{N}, \tag{4.1.1}$$

so that Proposition 4.1.1 reads

$$\sum_{p \le y} \frac{\chi(p) - 1}{p} \ll h(T).$$

Now Proposition 4.1.1 allows us to show that we can indeed approximate χ by 1 when performing logarithmic sums.

Proposition 4.1.2. *Suppose that $\chi \in A_{\pm}(T, N)$ and let $f(n)$ be any bounded function. Let $y > T$ be such that $\log\log y = \left(1 + O\left(\frac{1}{N}\right)\right) \log\log T$ and let $0 \le u \le u' < \exp((\log y)^{3/5-\epsilon})$. Then writing $w = \max\{0, u - 1\}$ and $w' = \max\{u' - u, u' - 1\}$ we have*

$$\left| \sum_{\substack{y^u \le n \le y^{u'} \\ P(n) \le y}} \frac{\chi(n)}{n} f(n) - \sum_{\substack{y^u \le n \le y^{u'} \\ P(n) \le y}} \frac{f(n)}{n} \right| \ll h(T) \log y \int_{w}^{w'} \rho(t) dt.$$

We will need the following lemmas.

Lemma 4.1.1. *Let $|\alpha| \le 1$, then*

$$|\alpha\beta - 1| \le |\beta - 1| + |\alpha - 1|.$$

Proof. Observe that

$$\begin{aligned}|\alpha\beta - 1| &\le |\alpha\beta - \alpha| + |\alpha - 1| \\ &\le |\beta - 1| + |\alpha - 1|.\end{aligned}$$

$\square$

As an immediate corollary, by complete multiplicativity of characters, we obtain

Corollary 4.1.1.
$$|\chi(n)-1| \le \sum_{p^k|n} |\chi(p)-1|$$

Lemma 4.1.2. *Let $y > T$ be such that $\log\log y = \left(1 + O\left(\frac{1}{N}\right)\right)\log\log T$ and assume that $\chi \in A_{\pm}(T, N)$. Then*
$$\sum_{\substack{p\le y\\ k\ge 1}} \frac{|\chi(p)-1|}{p^k} \ll h(T)$$

Proof. Since the sum over k is a geometric series, we have
$$\begin{aligned}\sum_{\substack{p\le y\\ k\ge 1}} \frac{|\chi(p)-1|}{p^k} &= \sum_{p\le y} \frac{|\chi(p)-1|}{p-1}\\ &\le 2\sum_{p\le y} \frac{|\chi(p)-1|}{p}\\ &\ll h(T)\end{aligned}$$
by Proposition 4.1.1. $\square$

Proof of Proposition 4.1.2. Start with
$$\left|\sum_{\substack{y^u\le n\le y^{u'}\\ P(n)\le y}} \frac{\chi(n)}{n} f(n) - \sum_{\substack{y^u\le n\le y^{u'}\\ P(n)\le y}} \frac{f(n)}{n}\right| \ll \sum_{\substack{y^u\le n\le y^{u'}\\ P(n)\le y}} \frac{|\chi(n)-1|}{n},$$
then using Corollary 4.1.1, we have
$$\begin{aligned}\sum_{\substack{y^u\le n\le y^{u'}\\ P(n)\le y}} \frac{|\chi(n)-1|}{n} &\le \sum_{\substack{y^u\le n\le y^{u'}\\ P(n)\le y}} \frac{1}{n}\sum_{p^k|n} |\chi(p)-1|\\ &= \sum_{\substack{p\le y\\ k\ge 1}} \frac{|\chi(p)-1|}{p^k} \sum_{\substack{\frac{y^u}{p^k}\le m\le \frac{y^{u'}}{p^k}\\ P(m)\le y}} \frac{1}{m}\\ &= \sum_{\substack{p\le y\\ k\ge 1}} \frac{|\chi(p)-1|}{p^k}\left(\log y\int_{u-v_p}^{u'-v_p} \rho(t)dt + O(\rho(u-v_p))\right)\end{aligned}$$

by Lemma 1.3.7, with $v_p = k\frac{\log p}{\log y}$. Now if $p^k \le y$, then $v_p \le \min\{u, 1\}$ and if $p^k > y$, then as $p \le y$, we must have $k \ge 2$ and therefore $y^{1/k} < p \le y$. So next we split the sum to cover these two cases.

$$\sum_{\substack{y^u \le n \le y^{u'} \\ P(n) \le y}} \frac{|\chi(n) - 1|}{n} \le \log y \left[\sum_{p^k \le y} \frac{|\chi(p) - 1|}{p^k} \left(\int_{\max\{0,u-1\}}^{\max\{u'-u,u'-1\}} \rho(t)dt + O\left(\frac{\rho(u-1)}{\log y}\right)\right)\right.$$
$$\left. + \sum_{k \ge 2} \sum_{y^{1/k} < p \le y} \frac{|\chi(p) - 1|}{p^k} \left(\int_{u-v_p}^{u'-v_p} \rho(t)dt + O\left(\frac{\rho(u-v_p)}{\log y}\right)\right)\right].$$

Bounding the second sum trivially, we have

$$\sum_{k \ge 2} \sum_{y^{1/k} < p \le y} \frac{|\chi(p) - 1|}{p^k} \left(\int_{u-v_p}^{u'-v_p} \rho(t)dt + O\left(\frac{\rho(u-v_p)}{\log y}\right)\right) \ll \int_2^{\log y} \int_{y^{1/k}}^{y} \frac{1}{t^k} dt$$
$$\ll \frac{\log y}{\sqrt{y}}.$$

and using Lemma 4.1.2 to bound the first sum, we get

$$\sum_{\substack{y^u \le n \le y^{u'} \\ P(n) \le y}} \frac{|\chi(n) - 1|}{n} \ll h(T) \log y \int_w^{w'} \rho(t)dt + \frac{\log^2 y}{\sqrt{y}},$$

where $w = \max\{0, u-1\}$ and $w' = \max\{u'-u, u'-1\}$. $\square$

The bound on the characters in $A_{\pm}(T, N)$ also allows us to evaluate logarithmic character sums over y-smooth numbers. So next we show

Proposition 4.1.3. *Assume that $\chi \in A_{\pm}(T, N)$, then for $y \ge T$ with $\log\log y = \left(1 + O\left(\frac{1}{N}\right)\right) \log\log T$ and $B < \exp((\log y)^{3/5-\epsilon})$, we have*

$$\sum_{\substack{n > y^B \\ P(n) \le y}} \frac{\chi(n)}{n} = \log y \int_B^{\infty} \rho(u)du + O(1 + h(T) \log y).$$

We start by writing the sum as

$$\sum_{\substack{n > y^B \\ P(n) \le y}} \frac{\chi(n)}{n} = \sum_{\substack{n \ge 1 \\ P(n) \le y}} \frac{\chi(n)}{n} - \sum_{\substack{n \le y^B \\ P(n) \le y}} \frac{\chi(n)}{n}, \tag{4.1.2}$$

and we first use Proposition 4.1.1 to evaluate the first sum on the right hand side of (4.1.2).

Lemma 4.1.3. *Assume that* $\chi \in A_{\pm}(T,N)$*, then for* $y \geq T$ *with* $\log\log y = \left(1+O\left(\frac{1}{N}\right)\right)\log\log T$*,*

$$\sum_{\substack{n\geq 1\\ P(n)\leq y}} \frac{\chi(n)}{n} = e^{\gamma}\log y + O\left(h(T)\log y\right).$$

Proof. Taking the Euler product, we have

$$\sum_{\substack{n\geq 1\\ P(n)\leq y}} \frac{\chi(n)}{n} = \prod_{p\leq y}\left(1-\frac{\chi(p)}{p}\right)^{-1}.$$

Consider,

$$\begin{aligned}\left|\frac{\prod_{p\leq y}\left(1-\frac{\chi(p)}{p}\right)^{-1}}{\prod_{p\leq y}\left(1-\frac{1}{p}\right)^{-1}}\right| &= \exp\left(\left|\sum_{p\leq y}\sum_{k\geq 1}\frac{\chi(p)^k-1}{kp^k}\right|\right)\\ &\leq \exp\left(\sum_{p\leq y}\sum_{k\geq 1}\frac{|\chi(p)^k-1|}{kp^k}\right).\end{aligned}$$

Applying Lemma 4.1.1 and computing the geometric series, we get, using the fact that $\max\limits_{p\leq T}|\chi(p)-1| \ll \frac{1}{N}$,

$$\begin{aligned}\left|\frac{\prod_{p\leq y}\left(1-\frac{\chi(p)}{p}\right)^{-1}}{\prod_{p\leq y}\left(1-\frac{1}{p}\right)^{-1}}\right| &\leq \exp\left(\sum_{p\leq y}\sum_{k\geq 1}\frac{|\chi(p)-1|}{p^k}\right)\\ &\leq \exp\left(\sum_{p\leq y}\frac{|\chi(p)-1|}{p-1}\right)\\ &\leq \exp(O(h(T)))\\ &= 1+O\left(h(T)\right).\end{aligned}$$

As this is also bounded below by 1, using Mertens estimate, we deduce that

$$\prod_{p\leq y}\left(1-\frac{\chi(p)}{p}\right)^{-1} = e^{\gamma}\log y + O\left(h(T)\log y\right).$$

□

Proof of Proposition 4.1.3. Starting with (4.1.2) and using Lemmas 4.1.3 and Proposition 4.1.2 with $f\equiv 1$, we get

$$\sum_{\substack{n>y^B \\ P(n)\le y}} \frac{\chi(n)}{n} = \sum_{\substack{n\ge 1 \\ P(n)\le y}} \frac{\chi(n)}{n} - \sum_{\substack{n\le y^B \\ P(n)\le y}} \frac{\chi(n)}{n}$$

$$= e^{\gamma} \log y + O\left(h(T)\log y\right) - \sum_{\substack{n\le y^B \\ P(n)\le y}} \frac{1}{n} + O\left(h(T)\log y \int_0^B \rho(u)du\right).$$

Now using Lemma 1.3.7, we have

$$\sum_{\substack{n>y^B \\ P(n)\le y}} \frac{\chi(n)}{n} = e^{\gamma}\log y - \log y \int_0^B \rho(u)du + O(1) + O(h(T)\log y)$$

$$= \log y \int_B^{\infty} \rho(u)du + O(1 + h(T)\log y),$$

where we deduced the last line from Lemma 1.3.8. ☐

4.2. Finding 1-pretentious characters: incursion in the world of lattices

It remains to show that we can find characters that belong to $A_{\pm}(T,N)$. In order to do so, we turn to our theorems on lattices from Chapter 3, which will provide us with the necessary tools to prove that there are a lot of characters taking value close to 1 on the small primes.

We start with the set containing even characters and we show the following bound which holds for all prime moduli q.

Proposition 4.2.1. *Let $N \ge 1$ and $T \ge 3$. Then*

$$|A_+(T,N)| \ge \frac{\phi(q)}{2N^{\pi(T)}}. \tag{4.2.1}$$

In particular, if q is prime, then

$$|A_+(T,N)| \gg \frac{q}{N^{2T/\log T}}. \tag{4.2.2}$$

Proof. Let $\theta_p = \theta_p(\chi) = \frac{\arg\left(\chi(p)\right)}{2\pi}$, and observe that

$$|\chi(p) - 1| = 2\pi|\theta_p| + O(\theta_p^2),$$

so that the bound in (4.0.1) is equivalent to

$$\max_{p\leq T}|\theta_p| \ll \frac{1}{N}.$$

That is, we are looking for a lower bound on the size of

$$C_+\left(\frac{1}{N},T\right) = \left\{\chi \pmod q : \chi(-1)=1, |\theta_p| \leq \frac{1}{N}\ \forall p \leq T\right\}.$$

So we let $k=\pi(T)$, we choose a generator χ for the group of characters and we consider the k-dimensional argument vector

$$V_\chi = (\theta_2, \theta_3, \ldots, \theta_{p_k}) \in (\mathbf{R}/\mathbf{Z})^k.$$

As the subgroup of even characters arises from taking χ^ℓ for even integers $1 \leq \ell \leq \phi(q)$, then each even character has an argument vector given by $\ell V_\chi \pmod 1$ for some even $1 \leq \ell \leq \phi(q)$.

Now, as χ has order $\phi(q)$ in the group of characters, then the lattice vector V_χ must have order d, where $d|\phi(q)$. However, since χ^ℓ produces distinct characters for each $1 \leq \ell \leq \phi(q)$, then for every integer $1 \leq \ell \leq d$, there must be $\frac{\phi(q)}{d}$ characters $\psi = \chi^r$ for which $rV_\chi \equiv_\chi \pmod 1$, and choosing to view each of these as distinct vectors and we may consider the vector V_χ to have order $M=\phi(q)$. That is, taking $\mathbf{u} = V_\chi$, by Corollary 3.1.1 from Chapter 3, we get that

$$\begin{aligned}\left|C_+\left(\frac{1}{N},T\right)\right| &= \left|C_{2^+}\left(\frac{1}{N},k\right)\right| \\ &\geq \frac{\phi(q)}{2N^k}.\end{aligned}$$

It follows that

$$\max_{p\leq T}|\chi(p)-1| \ll \frac{1}{N}$$

for at least $\frac{\phi(q)}{2N^k}$ even characters $\pmod q$, which proves the first part of the proposition. The second part of Proposition 4.2.1 is immediate.

□

For the set containing the odd characters, we obtain a slightly a weaker result which holds for most of the prime moduli q except for a small exceptional set. This limitation comes from our inability to exploit fully the Fourier analysis argument in Theorem 3 and improving this argument and removing the dependence on k in the upper bound for $|r_j|$ would lead to a result holding for all prime moduli q.

Proposition 4.2.2. *Let Q be a large integer and let $T \leq \frac{\log Q}{100}$ and $N \leq \frac{T}{2(\log T)^3}$. For all but at most $Q^{\frac{1}{10}}$ primes $q \leq Q$,*

$$|A_-(T, N)| \gg \frac{q}{N^{2T/\log T}}. \tag{4.2.3}$$

As in Proposition 4.2.1, the strategy to prove Lemma 4.2.2 will be to use our theorems on lattices from Chapter 3. In particular, the proposition will follow from Corollary 3.2.1 and in order to get the desired bound, we will be required to show that for most primes $q \leq Q$, there are no small vector $\mathbf{r} \in \mathbb{Z}^k$ such that $2(\mathbf{r} \cdot V_\chi) \equiv 0 \pmod 1$. This is the purpose of the following Lemma.

So again, let

$$\theta_{p_j} = \frac{\arg(\chi(p_j))}{2\pi}$$

and

$$\mathbf{V}_\chi(k) = (\theta_2, \theta_3, \cdots, \theta_{p_k})$$

Lemma 4.2.1. *Let Q be a large integer and $k \leq \frac{1}{60} \frac{\log Q}{\log \log Q}$. Let $\chi \pmod q$ be a character of order $q-1$ and let $\mathbf{u}_q = \mathbf{V}_\chi(k)$. For all but at most $Q^{\frac{1}{10}}$ primes $q \leq Q$, if $n(\mathbf{r} \cdot \mathbf{u}_q) \equiv 0$* mod 1*, for some integer $n \leq Q^{\frac{1}{160}}$, then there exists $j \leq k$ such that*

$$|r_j| > k^5.$$

Proof. For given prime q and $\chi \pmod q$ generating the group of character, let $\mathbf{u}_q = \mathbf{V}_\chi(k)$ be the argument vector. Define

$$S(Q) = \left\{ \frac{Q}{2} < q \leq Q : \exists \mathbf{r} \in \mathbb{Z}^k \text{ with } |r_j| \leq k^5 \text{ and } n(\mathbf{r} \cdot \mathbf{u}_q) \equiv 0 \mod 1,\, n \leq Q^{\frac{1}{160}} \right\}.$$

We will now show that the cardinality of $S(Q)$ is less than $Q^{\frac{1}{12}}$ which will allow us to show that for most primes q, the condition $n(\mathbf{r} \cdot \mathbf{u}_q) \equiv 0 \mod 1$ implies that the components of $\mathbf{r}$ are greater than k^5.

First, suppose that $q \in S(Q)$ and consider

$$\chi\left(\prod_{j\le k} p_j^{r_j n}\right) = \prod_{j\le k}\chi(p_j)^{r_j n}$$
$$= e^{2\pi i n(\mathbf{u}_q\cdot\mathbf{r})}$$
$$= e^0$$
$$= 1.$$

As χ is a generator for the group of characters, we deduce that $\prod_{j\le k} p_j^{r_j n} \equiv 1 \pmod{q}$, which means that

$$\prod_{r_j>0} p_j^{r_j n} \equiv \prod_{r_i<0} p_i^{|r_i| n} \pmod{q},$$

from which we deduce that

$$q \left| \prod_{r_j>0} p_j^{r_j n} - \prod_{r_i<0} p_i^{|r_i| n}. \right. \tag{4.2.4}$$

Now fixing n and r, we wish to count the number of primes for which (4.2.4) can hold. So let

$$s(r,n) = \#\left\{\frac{Q}{2} < q < Q : q \left| \prod_{r_j>0} p_j^{r_j n} - \prod_{r_i<0} p_i^{|r_i| n}\right.\right\},$$

and observe that

$$\prod_{q\in s(r,n)} q \left| \prod_{r_j>0} p_j^{r_j n} - \prod_{r_i<0} p_i^{|r_i| n}, \right.$$

so that

$$\prod_{q\in s(r,n)} q \le \left| \prod_{r_j>0} p_j^{r_j n} - \prod_{r_i<0} p_i^{|r_i| n}\right|.$$

Using the lower bound on q, we have that

$$\left(\frac{Q}{2}\right)^{\#s(r,n)} \le \left| \prod_{r_j>0} p_j^{r_j n} - \prod_{r_i<0} p_i^{|r_i| n}\right|$$
$$\le \prod_{j\le k} p_j^{|r_j| n}$$
$$\le \left(\prod_{j\le k} p_j\right)^{k^5 Q^{\frac{1}{160}}}$$
$$\le e^{k\log k(1+o(1))k^5 Q^{\frac{1}{160}}}$$
$$\le e^{2k^6 \log k Q^{\frac{1}{160}}}.$$

It follows that

$$\#s(r,n) \le \frac{2k^6 \log kQ^{\frac{1}{160}}}{\log\left(\frac{Q}{2}\right)}.$$

Now summing over all values of n and possible $\mathbf{r}$ we get that

$$\begin{aligned}
\#S(Q) &= \sum_{n,\mathbf{r}} s(r,n) \\
&\le \sum_{\substack{|r_j|\le k^5 \\ j\le k}} \sum_{n\le Q^{\frac{1}{160}}} k^6 \log kQ^{\frac{1}{160}} \\
&\le (2k^5+1)^k k^6 \log kQ^{\frac{1}{80}} \\
&\le (3k)^{5k} Q^{\frac{1}{80}}.
\end{aligned}$$

As $k \le \frac{1}{60}\frac{\log Q}{\log\log Q}$, we get that

$$\begin{aligned}
(3k)^{5k} &\le \left(\frac{3}{60}\frac{\log Q}{\log\log Q}\right)^{\frac{1}{12}\frac{\log Q}{\log\log Q}} \\
&\le \exp\left(\log\log Q \frac{\log Q}{12\log\log Q}\right) \\
&= Q^{\frac{1}{12}}.
\end{aligned}$$

Hence, putting this together, we obtain that

$$\begin{aligned}
\#S(Q) &\le Q^{\frac{1}{12}} Q^{\frac{1}{80}} \\
&\le Q^{\frac{23}{240}}.
\end{aligned}$$

Finally, to get the exceptional set of primes $q \le Q$, we write

$$S = \left\{q \le Q : \exists \mathbf{r} \in \mathbb{Z}^k \text{ with } |r_j| \le k^5 \text{ and } n(\mathbf{r}\cdot\mathbf{u}_q) \equiv 0 \mod 1,\ n \le Q^{\frac{1}{160}}\right\},$$

and summing we get that

$$
\begin{aligned}
S &= \sum_{l=0}^{\infty} S\left(\frac{Q}{2^l}\right) \\
&\leq \sum_{l=0}^{\infty} \left(\frac{Q}{2^l}\right)^{\frac{23}{240}} \\
&= Q^{\frac{23}{240}} \sum_{l=0}^{\infty} \left(\frac{1}{2^{\frac{23}{240}}}\right)^l \\
&\leq Q^{\frac{1}{10}},
\end{aligned}
$$

from which the lemma follows. □

With this restriction on the vector $\mathbf{r}$ at our disposition, we now prove Proposition 4.2.2.

Proof of Proposition 4.2.2. Let q be a prime and let $\mathbf{u}_q = \mathbf{V}_\psi(k) = (\theta_{p_1}, \cdots, \theta_{p_k})$ be the argument vector for ψ, where ψ is chosen to be a generator for the group of characters $\pmod q$. Because ψ has order $\phi(q) = q-1$ in the group of characters, we view $\mathbf{u}_q$ as a vector of order $q-1 = 2m$. Now, as in the even case, Proposition 4.2.2 is equivalent to finding a lower bound for

$$
C_-(\nu, T) = \left\{\chi \pmod q : \chi(-1) = -1, |\theta_p| \leq \nu, \forall p \leq T\right\},
$$

for $\nu \ll \frac{1}{N}$.

Letting $k = \pi(T)$ be the number of primes up to T, we observe that taking $d = 2$ as the divisor of the order $q-1 = 2m$, we have

$$
C_-\left(\nu, T\right) = C_{2^-}\left(\nu, k\right).
$$

That is, by Corollary 3.2.1, we have that

$$
\left|C_-\left(\frac{2}{N}, T\right)\right| \geq \frac{2m}{2N^k}
$$

provided that there are no vector $\mathbf{r} \in \mathbb{Z}^k$, with $|r_j| \leq k^4 N \log^2 N$ for all $j \leq k$, such that $2(\mathbf{r}\cdot\mathbf{u}_q) \equiv 0 \pmod 1$. But Lemma 4.2.1 states that for at all but at most $Q^{\frac{1}{10}}$ primes $q \leq Q$, the condition $2(\mathbf{r}\cdot\mathbf{u}_q) \equiv 0 \pmod 1$, implies that there is a $j \leq k$ for which $|r_j| > k^5$. As we chose $N \leq \frac{T}{2(\log T)^3}$, we have that

$$\begin{aligned} N \log^2 N &< \frac{T}{2(\log T)^3} \log^2 T \\ &= \frac{T}{2\log T} \\ &\leq \pi(T) \\ &= k \end{aligned}$$

It follows that $k^4 N \log^2 N < k^5$ and therefore, the conditions for Corollary 3.2.1 to hold are satisfied, and we conclude that for all of these primes q, we must indeed have that $\gg \frac{q-1}{2N^k}$ odd characters such that

$$|\chi(p) - 1| \ll |\theta_p| \ll \frac{1}{N}.$$

This proves the proposition. $\square$

Finding these 1-pretentious characters plays a key role in the proof of Theorem 1, as such characters will provide us large character sums. In the next chapter, we keep setting up the ground with some estimates for the proof of our main theorem in Chapter 6.

Chapter 5

Preliminary estimates

Before diving into the proof of Theorem 1, we gather in this chapter some estimates on exponential sums and smooth numbers that will be of use in Chapter 6. Although they may seem unmotivated for the time being, keep in mind that they are in preparation of the proof of our main theorem.

5.1. Some estimates on exponential sums

What stands out when investigating logarithmic exponential sums of the form

$$\sum_{n\in I}\frac{e(\pm\alpha n)}{n} \tag{5.1.1}$$

is that all the action occurs when n is around $\frac{1}{\alpha}$. As we will see, this will have a direct impact on the logarithmic character sums that we evaluate in Theorem 1.

We start with a technical lemma that will allow us to handle the error terms in Lemma 5.1.2 and Lemma 5.1.4.

Lemma 5.1.1. *Let $\alpha \in (0,1)$ and let $Y \geq 1$, then*

$$\int_Y^\infty \frac{\{t\}e(\pm\alpha t)}{t}dt \ll 1+\frac{1}{\alpha Y}$$

PROOF.

$$\begin{aligned}\int_Y^\infty \frac{\{t\}e(\pm\alpha t)}{t}dt &= \sum_{n\geq\lfloor Y\rfloor}\int_0^1 \frac{te(\pm\alpha(t+n))}{t+n}dt + O\left(\frac{1}{Y}\right)\\ &= \int_0^1 te(\pm\alpha t)\left(\sum_{n\geq\lfloor Y\rfloor}\frac{e(\pm\alpha n)}{t+n}\right)dt + O\left(\frac{1}{Y}\right).\end{aligned}$$

Observe that

$$\left|\frac{e(\pm\alpha n)}{t+n}-\frac{e(\pm\alpha n)}{n}\right|\leq\left|\frac{1}{n+1}-\frac{1}{n}\right|\leq\frac{1}{n^2},$$

and therefore

$$\int_Y^\infty\frac{\{t\}e(\pm\alpha t)}{t}dt=\int_0^1 te(\pm\alpha t)dt\left(\sum_{n\geq\lfloor Y\rfloor}\frac{e(\pm\alpha n)}{n}+O\left(\frac{1}{n^2}\right)\right).$$

Now it is not hard to see that the integral on the right hand side is bounded by 1 and by partial summation, we have that

$$\begin{aligned}\sum_{n\geq\lfloor Y\rfloor}\frac{e(\pm\alpha n)}{n}&=\int_Y^\infty\sum_{n\leq t}e(\pm\alpha n)\frac{dt}{t^2}+O(1)\\&=\int_Y^\infty\frac{e(\pm\alpha(\lfloor t\rfloor+1))-1}{e(\pm\alpha)-1}\frac{dt}{t^2}+O(1)\\&\ll\frac{1}{\alpha}\int_Y^\infty\frac{1}{t^2}dt+1\\&\ll\frac{1}{\alpha Y}+1.\end{aligned}$$

Putting this together, it follows that

$$\int_Y^\infty\frac{\{t\}e(\pm\alpha t)}{t}dt\ll 1+\frac{1}{\alpha Y}.$$

□

The next lemma emphasizes that most contributions to (5.1.1) happen around $\frac{1}{\alpha}$ by showing that the tail of the sum is negligeable.

Lemma 5.1.2. *Let $\alpha\in(0,1)$, then*

$$\sum_{n\geq\frac{1}{\alpha}}\frac{e(\pm\alpha n)}{n}=\sum_{\frac{1}{\alpha}<n\leq\frac{|\log\alpha|^c}{\alpha}}\frac{e(\pm\alpha n)}{n}+O\left(\frac{1}{|\log\alpha|^c}\right).$$

Proof.

$$\sum_{n\ge\frac{1}{\alpha}}\frac{e(\pm\alpha n)}{n}-\sum_{\frac{1}{\alpha}\le n\le\frac{|\log\alpha|^c}{\alpha}}\frac{e(\pm\alpha n)}{n}=\sum_{n>\frac{|\log\alpha|^c}{\alpha}}\frac{e(\pm\alpha n)}{n}$$
$$=\int_{\frac{|\log\alpha|^c}{\alpha}}^{\infty}\frac{e(\pm\alpha t)}{t}d(t-\{t\})$$
$$=\int_{\frac{|\log\alpha|^c}{\alpha}}^{\infty}\frac{e(\pm\alpha t)}{t}dt-\int_{\frac{|\log\alpha|^c}{\alpha}}^{\infty}\frac{e(\pm\alpha t)}{t}d\{t\}$$
$$=\int_{\frac{|\log\alpha|^c}{\alpha}}^{\infty}\frac{e(\pm\alpha t)}{t}dt\mp 2\pi i\alpha\int_{\frac{|\log\alpha|^c}{\alpha}}^{\infty}\frac{\{t\}e(\pm\alpha t)}{t}dt+O\left(\frac{\alpha}{|\log\alpha|^c}\right).$$

Now the second term is $O(\alpha)$ by Lemma 5.1.1 and noticing that

$$\int_n^{n+1}\frac{e(\pm w)}{w}dw=\int_n^{n+1}\frac{e(\pm w)}{n}dw+\int_n^{n+1}e(\pm w)\left(\frac{1}{w}-\frac{1}{n}\right)dw$$
$$=\frac{1}{n}\int_0^1 e(\pm w)dw+\int_n^{n+1}e(\pm w)\left(\frac{n-w}{nw}\right)dw$$
$$\ll\frac{1}{n^2}$$

allows us to deduce that

$$\int_{\frac{|\log\alpha|^c}{\alpha}}^{\infty}\frac{e(\pm\alpha t)}{t}dt=\int_{|\log\alpha|^c}^{\infty}\frac{e(\pm w)}{w}dw$$
$$\ll\sum_{n=|\log\alpha|^c-1}^{\infty}\frac{1}{n^2}$$
$$\ll\frac{1}{|\log\alpha|^c}.$$

□

Analogously, it is easy to see that the beginning of the following sum does not contribute too much.

Lemma 5.1.3.

$$\sum_{n\le\frac{1}{\alpha}}\frac{1-e(\pm\alpha n)}{n}=\sum_{\frac{1}{\alpha|\log\alpha|}<n\le\frac{1}{\alpha}}\frac{1-e(\pm\alpha n)}{n}+O\left(\frac{1}{|\log\alpha|}\right)$$

Proof.

$$\sum_{n\le\frac{1}{\alpha|\log\alpha|}}\frac{1-e(\pm\alpha n)}{n}=O\left(\sum_{n\le\frac{1}{\alpha|\log\alpha|}}\frac{\alpha n}{n}\right)$$
$$=O\left(\alpha\frac{1}{\alpha|\log\alpha|}\right)$$
$$=O\left(\frac{1}{|\log\alpha|}\right).$$

$\square$

Interestingly, putting the sums in Lemma 5.1.3 and Lemma 5.1.2 together gives rise to a constant. This will play an important role for the proof of Theorem 2.

Lemma 5.1.4. *Let $\alpha\in(0,1)$, then*

$$\sum_{n\le\frac{1}{\alpha}}\frac{1-e(\pm\alpha n)}{n}-\sum_{n>\frac{1}{\alpha}}\frac{e(\pm\alpha n)}{n}=\log(2\pi)+\gamma\mp\frac{i\pi}{2}+O\left(\alpha|\log\alpha|\right).$$

Proof. We have

$$\begin{aligned}\sum_{n\le\frac{1}{\alpha}}\frac{1-e(\pm\alpha n)}{n}-\sum_{n>\frac{1}{\alpha}}\frac{e(\pm\alpha n)}{n}&=\int_1^{\frac{1}{\alpha}}\frac{1-e(\pm\alpha t)}{t}d\lfloor t\rfloor-\int_{\frac{1}{\alpha}}^{\infty}\frac{e(\pm\alpha t)}{t}d\lfloor t\rfloor\\&=\int_1^{\frac{1}{\alpha}}\frac{1-e(\pm\alpha t)}{t}d(t-\{t\})-\int_{\frac{1}{\alpha}}^{\infty}\frac{e(\pm\alpha t)}{t}d(t-\{t\})\\&=\int_1^{\frac{1}{\alpha}}\frac{1}{t}dt-\int_1^{\infty}\frac{e(\pm\alpha t)}{t}dt+\int_1^{\frac{1}{\alpha}}\frac{1-e(\pm\alpha t)}{t}d(\{t\})-\int_{\frac{1}{\alpha}}^{\infty}\frac{e(\pm\alpha t)}{t}d(\{t\}).\end{aligned}\tag{5.1.2}$$

Now, by integrating by parts the third integral and noting that $|1-e(\alpha t)|\ll\alpha t$ for $t<\frac{1}{\alpha}$, we have that

$$\begin{aligned}\int_1^{\frac{1}{\alpha}}\frac{1-e(\pm\alpha t)}{t}d(\{t\})&=\{t\}\frac{1-e(\pm\alpha t)}{t}\bigg|_1^{\frac{1}{\alpha}}+\int_1^{\frac{1}{\alpha}}\{t\}\left(\frac{\pm2\pi i\alpha e(\pm\alpha t)}{t}+\frac{1-e(\pm\alpha t)}{t^2}\right)dt\\&\ll\alpha+\alpha\int_1^{\frac{1}{\alpha}}\frac{1}{t}dt+\int_1^{\frac{1}{\alpha}}\frac{\alpha t}{t^2}dt\\&\ll\alpha|\log\alpha|.\end{aligned}$$

Similarly, integrating by parts the last integral in (5.1.2), we have

$$\int_{\frac{1}{\alpha}}^{\infty} \frac{e(\pm\alpha t)}{t} d(\{t\}) = \mp 2\pi i\alpha \int_{\frac{1}{\alpha}}^{\infty} \frac{\{t\}e(\pm\alpha t)}{t} dt + O\left(\alpha\right),$$

and by Lemma 5.1.1 with $Y = \frac{1}{\alpha}$, the integral is $\ll 1$, and we obtain

$$\int_{\frac{1}{\alpha}}^{\infty} \frac{e(\pm\alpha t)}{t} d(\{t\}) \ll \alpha.$$

Going back to (5.1.2), in which we rewrite the exponential integral as sine and cosine integrals, we obtain

$$\begin{aligned}\sum_{n\leq\frac{1}{\alpha}} \frac{1-e(\pm\alpha n)}{n} - \sum_{n>\frac{1}{\alpha}} \frac{e(\pm\alpha n)}{n} &= \int_1^{\frac{1}{\alpha}} \frac{1}{t} dt - \int_1^{\infty} \frac{e(\pm\alpha t)}{t} dt + O\left(\alpha|\log\alpha|\right) \\ &= \log\left(\frac{1}{\alpha}\right) - \left(\int_{2\pi\alpha}^{\infty} \frac{\cos t}{t} dt \pm i \int_{2\pi\alpha}^{\infty} \frac{\sin t}{t} dt\right) + O\left(\alpha|\log\alpha|\right).\end{aligned}$$

The cosine integrals can be estimated using the Taylor expansions and referring to [**9**] p.(106), we know that

$$-\int_x^{\infty} \frac{\cos t}{t} dt = \gamma + \log x + \sum_{k=1}^{\infty} \frac{(-x^2)^k}{2k(2k)!},$$

hence we deduce that

$$-\int_{2\pi\alpha}^{\infty} \frac{\cos t}{t} dt = \gamma + \log(2\pi\alpha) + O(\alpha^2).$$

Now it is easily seen, using the Taylor series for sine, that

$$\int_{2\pi\alpha}^{\infty} \frac{\sin t}{t} dt = \int_0^{\infty} \frac{\sin t}{t} dt + O(\alpha),$$

and it is known (see for example [**1**] p.232) that

$$\int_0^{\infty} \frac{\sin t}{t} dt = \frac{\pi}{2}.$$

Putting this together, we reach the conclusion that

$$\begin{aligned}\sum_{n\leq\frac{1}{\alpha}}\frac{1-e(\pm\alpha n)}{n}-\sum_{n>\frac{1}{\alpha}}\frac{e(\pm\alpha n)}{n}&=\log\left(\frac{1}{\alpha}\right)+\log(2\pi\alpha)+\gamma\mp\frac{i\pi}{2}+O\left(\alpha|\log\alpha|\right)\\&=\log(2\pi)+\gamma\mp\frac{i\pi}{2}+O\left(\alpha|\log\alpha|\right),\end{aligned}$$

as desired. □

5.2. Some estimates on smooth numbers

We start this section with an estimate showing that the tail of a logarithmic sum over y-smooth integers is small. This will help us bound the error term in the proof of Theorem 1. The argument follows the proof of Lemma 3.2 in [**2**].

Lemma 5.2.1. *Let $y\geq 100$, then*

$$\sum_{\substack{n>y^{\log\log y}\\P(n)\leq y}}\frac{1}{n}\ll\frac{1}{(\log y)^{\log_3 y-3/2}},$$

where the

Proof. We have

$$\sum_{\substack{n>y^{\log\log y}\\P(n)\leq y}}\frac{1}{n}\ll\sum_{\substack{y^{\log\log y}<n\leq y^{\sqrt{\log y}}\\P(n)\leq y}}\frac{1}{n}+\sum_{\substack{n\geq y^{\sqrt{\log y}}\\P(n)\leq y}}\frac{1}{n}$$

For the first sum of the right hand side, we use Lemma 1.3.7 and Lemma 1.3.2 to get

$$\begin{aligned}\sum_{\substack{y^{\log\log y}<n\leq y^{\sqrt{\log y}}\\P(n)\leq y}}\frac{1}{n}&\ll\log y\int_{\log\log y}^{\sqrt{\log y}}\rho(u)du+\rho(\log\log y)\\&\ll(\log y)^{3/2}\rho(\log\log y)\\&\ll\frac{\log y\sqrt{\log y}}{(\log\log y)^{\log\log y}}\\&\ll\frac{1}{(\log y)^{\log_3 y-3/2}}\end{aligned}$$

For the second sum, given $\epsilon = \frac{1}{\log y}$, we have

$$\begin{aligned}
\sum_{\substack{n \geq y^{\sqrt{\log y}} \\ P(n) \leq y}} \frac{1}{n} &\leq \sum_{\substack{n \geq y^{\sqrt{\log y}} \\ P(n) \leq y}} \frac{1}{n} \left(\frac{n}{y^{\sqrt{\log y}}} \right)^{\epsilon} \\
&\leq e^{-\sqrt{\log y}} \sum_{P(n) \leq y} \frac{1}{n^{1-\epsilon}} \\
&\leq e^{-\sqrt{\log y}} \prod_{p \leq y} \left(1 - \frac{1}{p^{1-\epsilon}} \right)
\end{aligned}$$

As $p^{\epsilon} = 1 + O\left(\frac{\log p}{\log y}\right)$ for $p \leq y$, we have

$$\begin{aligned}
\sum_{p \leq y} \frac{1}{p^{1-\epsilon}} - \sum_{p \leq y} \frac{1}{p} &\ll \sum_{p \leq y} \frac{1}{p} \left(\frac{\log p}{\log y} \right) \\
&\ll 1
\end{aligned}$$

and thus, for y large enough, putting this together we deduce that

$$\sum_{\substack{y^{\log \log y} < n \leq z \\ P(n) \leq y}} \frac{\chi(n) e(n\alpha)}{n} \ll \frac{1}{(\log y)^{\log_3 y - 3/2}}.$$

$\square$

Even though smooth numbers are often major allies in evaluating sums over integers, they can also be an obstacle to our ability to evaluate sums. The following lemma shows that on small intervals, the smoothness condition can be removed.

Lemma 5.2.2. *Let $y \geq 2$ and let $f(t)$ be a differentiable bounded function on any interval $I \subset \left[\frac{y^B}{\log y}, y^B (\log y)^c\right]$, then for $B < \exp((\log y)^{3/5-\epsilon})$ and $c \geq 0$ we have*

$$\sum_{\substack{n \in I \\ P(n) \leq y}} \frac{f(n)}{n} = \rho(B) \sum_{n \in I} \frac{f(n)}{n} + O\left(\frac{\rho(B) \log(B+1) (\log \log y)^2}{\log y} \right).$$

Proof. Let I be any subinterval of $\left[\frac{y^B}{\log y}, y^B(\log y)^c\right]$. By partial summation we have

$$\begin{aligned}
\sum_{\substack{n\in I\\ P(n)\le y}} \frac{f(n)}{n} &= \int_I \frac{f(t)}{t} d(\psi(t,y)) \\
&= \int_I \frac{f(t)}{t} d\left(t\rho(u)\left(1+O\left(\frac{\log u}{\log y}\right)\right)\right).
\end{aligned}$$

Now for t in that range we have that $\log u = O(\log B)$ and by Lemma 1.3.5, $\rho(u) = \rho(B) + O\left(\frac{\rho(B)\log(B+1)\log\log y}{\log y}\right)$, therefore

$$\sum_{\substack{n\in I\\ P(n)\le y}} \frac{f(n)}{n} = \left(\rho(B) + O\left(\frac{\rho(B)\log(B+1)\log\log y}{\log y}\right)\right) \int_I \frac{f(t)}{t} dt.$$

On the other hand, using partial summation again, we have

$$\begin{aligned}
\sum_{n\in I} \frac{f(n)}{n} &= \frac{f(t)}{t}(t+O(1))\Big|_I - \int_I \left(\frac{f'(t)}{t} - \frac{f(t)}{t^2}\right)(t+O(1))dt \\
&= f(t)\Big|_I + \int_I \frac{f(t)}{t} - f'(t)dt + O\left(\frac{(\log\log y)^2}{\log y}\right) \\
&= f(t)\Big|_I - f(t)\Big|_I + \int_I \frac{f(t)}{t}dt + O\left(\frac{(\log\log y)^2}{\log y}\right) \\
&= \int_I \frac{f(t)}{t}dt + O\left(\frac{(\log\log y)^2}{\log y}\right).
\end{aligned}$$

Hence comparing both sides, we deduce that

$$\begin{aligned}
\sum_{\substack{n\in I\\ P(n)\le y}} \frac{f(n)}{n} &= \left(\rho(B) + O\left(\frac{\rho(B)\log(B+1)\log\log y}{\log y}\right)\right)\left(\sum_{n\in I}\frac{f(n)}{n} + O\left(\frac{(\log\log y)^2}{\log y}\right)\right) \\
&= \rho(B)\sum_{n\in I}\frac{f(n)}{n} + O\left(\frac{\rho(B)\log(B+1)(\log\log y)^2}{\log y}\right),
\end{aligned}$$

which ends the proof of the lemma ∎

As we undergo the proof of Theorem 2, we will have to face such a sum and Lemma 5.2.2 will come in handy. We are now ready for the proof of our main theorem.

Chapter 6

Proof of the main theorem

In this chapter, we undertake the main task of this thesis, that of proving Theorem 1. In the following, we assume that q is a large prime and we aim to investigate the maximal value of character sums modulo q as the length of the summation gets close to q. Namely, we wish to find a lower bound for

$$\max_{\chi \neq \chi_0} \left| \sum_{n \leq \frac{q}{(\log q)^B}} \chi(n) \right|. \tag{6.0.1}$$

Remark. *In order to obtain our main theorem, in section 6.3.2, we will restrict our choice of primes q, which will make for the exceptional set in Theorem 1.*

For easier reference, we take the time to restate Theorem 1 here.

Theorem 1 *Let Q be a large integer, for all but at most $Q^{\frac{1}{10}}$ primes $q \leq Q$, if $1 \leq B < \frac{\log\log\log q}{\log\log\log\log q}$, then*

$$\max_{\chi \neq \chi_0} \left| \sum_{n \leq \frac{q}{(\log q)^B}} \chi(n) \right| \geq \frac{\sqrt{q}}{\pi} \log\log q \int_B^\infty \rho(u)du + O(\sqrt{q}\log\log\log q).$$

To prove Theorem 1, we show that there exists an odd character for which such a bound holds, so we stress the fact that the maximum is actually arising from odd characters. For even characters, we restate the bound that we obtain and observe again that, although it is weaker, it holds for all prime moduli q.

Theorem 2 *Let q be any large prime and let $1 \le B < \frac{\log\log\log q}{\log\log\log\log q}$, then*

$$\max_{\substack{\chi \neq \chi_0 \\ \chi \text{ even}}} \left| \sum_{n \le \frac{q}{(\log q)^B}} \chi(n) \right| \ge \frac{\rho(B)}{2}\sqrt{q} + O\left(\frac{\rho(B)\log(B+1)\sqrt{q}(\log\log\log q)^2}{\log\log q} \right).$$

6.1. Set up of the problem

In the following, we let $y = \log q$, $\alpha = \frac{1}{y^B}$ for some $1 \le B \le \frac{\log\log\log}{\log\log\log\log q}$ and we let $z = q^{11/21}$.

We start by rewriting the sum using Pólya's Fourier expansion 0.4.1, which will allow us to evaluate the sum when the character sum ranges over $n \le \frac{q}{(\log q)^B}$. That is, we have

$$\sum_{n \le \alpha q} \chi(n) = \frac{\tau(\chi)}{2\pi i} \sum_{1 \le |n| \le z} \overline{\chi}(n) \frac{1 - e(-\alpha n)}{n} + O\left(\frac{q \log q}{z} \right), \tag{6.1.1}$$

where $|\tau(\chi)| = \sqrt{q}$ by Theorem 2.1.2.
As a consequence, we wish to study the behavior of

$$\sum_{1 \le n \le z} \frac{\chi(n)}{n} \left(1 - e(\pm \alpha n)\right).$$

Now our goal is to find a character for which the sum gets big, so in order to get better control on the sum, we focus on a set of characters whose sum can be restricted to y-smooth numbers. That is, for $1 \le y \le z$ we let $\delta \in \left[\frac{1}{\log y}, 1\right]$ and define

$$A_\delta = \left\{ \chi \pmod q : \left| \sum_{\substack{1 \le |n| \le z \\ P(n) > y}} \frac{\chi(n)}{n} \left(1 - e(-\alpha n)\right) \right| \le e^\gamma \delta \right\}. \tag{6.1.2}$$

We believe that the bound in (6.1.2) should hold for all characters modulo q, for q large enough, so we recall the slightly stronger conjecture

Conjecture 1 *Let q be large and let χ be a character modulo q. Let $\log q \le y \le x$, then*

$$\max_{\alpha \in [0,1]} \left| \sum_{\substack{n \le \\ P(n) > y}} \frac{\chi(n)}{n} e(\alpha n) \right| \ll 1.$$

We note that if Conjecture 1 holds then the proof shows that Theorem 1 is best possible for most prime moduli q, as the inequality sign then becomes an equality sign. However, for the purpose of our proof, we will use Theorem 4.2 in [**2**], from which we know that

$$\#\{\chi \pmod q : \chi \notin A_\delta\} \ll q^{1-\frac{\delta^2}{\log\log q}} + q^{1-\frac{1}{500\log\log q}}. \tag{6.1.3}$$

If we were to consider only the case of odd characters, we could take $\delta = 1$ without any problems. However, because the main term in Theorem 2 is much smaller for even character case, we have to be a little more delicate with the choice of δ so that the error term doesn't get too big. That is, in order to obtain a satisfying result for the even characters, we will later choose δ to be of size $\frac{\log\log y}{\sqrt{\log y}}$.

We now restrict our attention to characters in A_δ, which means that we can focus on the sum

$$\sum_{\substack{1\le n\le z\\ P(n)\le y}} \frac{\chi(n)}{n}\left(1 - e(\pm\alpha n)\right).$$

Careful analysis leads us to split the sum in the following way, so that the main contribution in the sum is brought to light. Write

$$\sum_{\substack{1\le n\le z\\ P(n)\le y}} \frac{\chi(n)}{n}\left(1 - e(\pm\alpha n)\right) = S_1 + S_2^{\pm} + S_3^{\pm}, \tag{6.1.4}$$

where the sum

$$S_1 = \sum_{\substack{n\ge y^B\\ P(n)\le y}} \frac{\chi(n)}{n}$$

will give the main contribution in the odd character case, the sum

$$S_2^{\pm} = \sum_{\substack{\frac{y^B}{\log y}\le n\le y^B\\ P(n)\le y}} \chi(n)\frac{1-e(\pm\alpha n)}{n} - \sum_{\substack{y^B\le n\le y^B(\log y)^5\\ P(n)\le y}} \chi(n)\frac{e(\pm\alpha n)}{n}$$

will give the main term in the even character case, and finally

$$S_3^{\pm} = \sum_{\substack{1 \le n \le \frac{y^B}{\log y} \\ P(n) \le y}} \chi(n) \frac{1 - e(\pm \alpha n)}{n} - \sum_{\substack{y^B (\log y)^5 \le n \le y^{\log \log y} \\ P(n) \le y}} \chi(n) \frac{e(\pm \alpha n)}{n} - \sum_{\substack{y^{\log \log y} \le n \le z \\ P(n) \le y}} \chi(n) \frac{e(\pm \alpha n)}{n} - \sum_{\substack{n \ge z \\ P(n) \le y}} \frac{\chi(n)}{n}$$

will contribute the error term.

6.2. S_3 : Ranges with small contribution

In this section, we dissect $S_3^{\pm}$ to show that it provides only a small contribution to 6.1.4. Starting from

$$S_3^{\pm} = \sum_{\substack{1 \le n \le \frac{y^B}{\log y} \\ P(n) \le y}} \chi(n) \frac{1 - e(\pm \alpha n)}{n} - \sum_{\substack{y^B (\log y)^5 \le n \le y^{\log \log y} \\ P(n) \le y}} \chi(n) \frac{e(\pm \alpha n)}{n} - \sum_{\substack{y^{\log \log y} \le n \le z \\ P(n) \le y}} \chi(n) \frac{e(\pm \alpha n)}{n} - \sum_{\substack{n \ge z \\ P(n) \le y}} \frac{\chi(n)}{n}, \tag{6.2.1}$$

we show that

Proposition 6.2.1. *Let $S_3^{\pm}$ be defined as in (6.2.1), then for y large enough, $1 \le B < \exp((\log y)^{3/5-\epsilon})$, then*

$$S_3^{\pm} \ll \frac{\sqrt{B}}{\log y}$$

In order to prove this, we treat of the sums in $S_3^{\pm}$ one at a time, Lemma 6.2.1 dealing with the first sum, Lemma 6.2.2 handling the second sum and the last 2 sums following from Lemma 5.2.1 in Lemma 6.2.3.

First we have

Lemma 6.2.1. *Let $y \ge 2$, $B < \exp((\log y)^{3/5-\epsilon})$ and let $\alpha = \frac{1}{y^B}$, then*

$$\sum_{\substack{1 \le n \le \frac{y^B}{\log y} \\ P(n) \le y}} \chi(n) \frac{1 - e(\pm \alpha n)}{n} \ll \frac{\rho(B)}{\log y}.$$

Proof. First, notice that as $\alpha = \frac{1}{y^B}$, we have $|\alpha n| < 1$ and thus

$$
\begin{aligned}
\frac{1-e(\pm\alpha n)}{n} &\ll \frac{\alpha n}{n} \\
&= \alpha
\end{aligned}
$$

Therefore, as each $|\chi(n)| \leq 1$,

$$
\begin{aligned}
\sum_{\substack{1\leq n\leq \frac{y^B}{\log y}\\ P(n)\leq y}} \chi(n)\frac{1-e(\pm\alpha n)}{n} &\ll \alpha \sum_{\substack{n\leq \frac{y^B}{\log y}\\ P(n)\leq y}} 1 \\
&\ll \frac{\rho\left(B-\frac{\log\log y}{\log y}\right)}{\log y} \\
&\ll \frac{\rho(B)}{\log y}
\end{aligned}
$$

by Lemma 1.3.5.

$\square$

The second sum in $S_3^{\pm}$ requires the use of a result from De la Bretèche for exponential sums with multiplicative coefficients over smooth numbers [**3**]. We obtain

Lemma 6.2.2. *Let $y \geq 2$ and let $B \geq 1$. Let also $c \geq 5$, then for $\alpha = \frac{1}{y^B}$,*

$$
\sum_{\substack{y^B(\log y)^c\leq n\leq y^{\log\log y}\\ P(n)\leq y}} \frac{\chi(n)e(\pm\alpha n)}{n} \ll \frac{\sqrt{B}}{\log y}.
$$

In order to prove Lemma 6.2.2, we use the following lemma which appears as Proposition 1 in [**3**].

Theorem 6.2.1. *Let $f(n)$ be a multiplicative function with $\sum_{n\leq t}|f(n)|^2 \leq A^2 t$, and suppose that there is $(a,m)=1$ such that $|\alpha-\frac{a}{m}| \leq \frac{1}{m^2}$ then*

$$
\sum_{\substack{n\leq x\\ P(n)\leq y}} f(n)e(\alpha n) \ll A^2 x\sqrt{\log x}\log y\left(\frac{\sqrt{y}}{\sqrt{x}}+\frac{\sqrt{m}}{\sqrt{x}}+\frac{1}{\sqrt{m}}+e^{-\sqrt{\log x}}\right).
$$

Corollary 6.2.1. *Let $\alpha = \frac{1}{y^B}$ for $B \geq 1$ and m be the closest integer to y^B. Write $x = y^{B+v}$ for $v \geq \frac{c \log\log y}{\log y}$. Then if $v \leq B$,*

$$\sum_{\substack{n \leq x \\ P(n) \leq y}} f(n) e(\pm \alpha n) \ll A^2 x \sqrt{\log x} \log y \left(\frac{\sqrt{m}}{\sqrt{x}} + e^{-\sqrt{\log x}} \right),$$

whereas if $v > B$ then

$$\sum_{\substack{n \leq x \\ P(n) \leq y}} f(n) e(\pm \alpha n) \ll A^2 x \sqrt{\log x} \log y \frac{1}{\sqrt{m}}.$$

Remark. *We wish to stress the fact that the limitation on the lower bound for B in Theorem 1 comes from Theorem 6.2.1, as taking $B < 1$ would lead to $\frac{\sqrt{y}}{\sqrt{x}} > 1$, which would make it the main error term and wouldn't allow us to bound that range appropriately.*

A simple use of partial summation and the results just stated allow us to deduce Lemma 6.2.2.

Proof of Lemma 6.2.2. Given $\alpha = \frac{1}{y^B}$, taking m to be the closest integer to y^B, we can apply Theorem 6.2.1 with $A = 1$.

That is, we have

$$\begin{aligned}
\sum_{\substack{y^B (\log y)^c \leq n \leq y^{\log\log y} \\ P(n) \leq y}} \frac{\chi(n) e(\pm \alpha n)}{n} &= \frac{1}{y^{\log\log y}} \sum_{\substack{n \leq y^{\log\log y} \\ P(n) \leq y}} \chi(n) e(\pm \alpha n) - \frac{1}{y^B (\log y)^c} \sum_{\substack{n \leq y^B (\log y)^c \\ P(n) \leq y}} \chi(n) e(\pm \alpha n) \\
&\quad + \int_{y^B (\log y)^c}^{y^{\log\log y}} \sum_{\substack{n \leq t \\ P(n) \leq y}} \chi(n) e(\pm \alpha n) \frac{dt}{t^2} \\
&\ll \frac{\sqrt{\log\log y} (\log y)^{3/2}}{y^{B/2}} + \frac{\sqrt{B}}{(\log y)^{\frac{c}{2} - \frac{3}{2}}} \\
&\quad + y^{B/2} \log y \int_{y^B (\log y)^c}^{y^{2B}} \frac{\sqrt{\log t}}{t^{3/2}} dt + \log y \int_{y^B (\log y)^c}^{y^{2B}} \frac{\sqrt{\log t}}{t} e^{-\sqrt{\log t}} dt \\
&\quad + \frac{\log y}{y^{B/2}} \int_{y^{2B}}^{y^{\log\log y}} \frac{\sqrt{\log t}}{t} dt.
\end{aligned}$$

Computing the integrals gives

$$\sum_{\substack{y^B(\log y)^c \le n \le y^{\log\log y} \\ P(n)\le y}} \frac{\chi(n)e(\pm\alpha n)}{n} \ll \frac{\sqrt{B}}{(\log y)^{\frac{c}{2}-\frac{3}{2}}} + Be^{-\sqrt{B\log y}}\log^2 y + \frac{\log y}{y^{B/2}}(\log y \log\log y)^{3/2}$$
$$\ll \frac{\sqrt{B}}{\log y},$$

whenever $c \ge 5$. $\square$

The next lemma deals with the two last sums of (6.2.1) and follows directly from Lemma 5.2.1.

Lemma 6.2.3. *Let χ be a character modulo q, let α be any real number in $(0,1]$ and $z \ge y^{\log\log y}$, then*

$$\sum_{\substack{y^{\log\log y}<n\le z \\ P(n)\le y}} \frac{\chi(n)e(\pm\alpha n)}{n} + \sum_{\substack{n\ge z \\ P(n)\le y}} \frac{\chi(n)}{n} \ll \frac{1}{(\log y)^{\log_3 y - 3/2}}.$$

Finally, putting Lemmas 6.2.1, 6.2.3 and 6.2.2 together gives Proposition 6.2.1.

6.3. S_1 and S_2: the main contributions

Our strategy in order to evaluate S_1 and S_2 will be to use characters that pretends to be 1, so that $\chi \in A_\pm(N,T)$, where we recall that

$$A_\pm(N,T) = \left\{\chi \pmod q : \chi(-1) = \pm 1 \ , \max_{p\le T}|\chi(p)-1| \ll \frac{1}{N} \ \forall p \le T\right\}.$$

This supposes that our choice of character will satisfy

$$\max_{p\le T}|\chi(p)-1| \ll \frac{1}{N},$$

and using this hypothesis brings us back to the results we derived in Chapter 4.

First, as a consequence of Proposition 4.1.3, we can evaluate S_1 directly, getting

Proposition 6.3.1. *Let χ be in $A_\pm(N,T)$, then for $y \ge T$ with $\log\log y = \left(1+\frac{1}{N}\right)\log\log T$ and $B < \exp((\log y)^{3/5-\epsilon})$, we have*

$$S_1 = \sum_{\substack{n>y^B \\ P(n)\le y}} \frac{\chi(n)}{n} = \log y \int_B^\infty \rho(u)du + O(1 + h(T)\log y).$$

This constitutes our main term in Theorem 1 and it remains to evaluate S_2.

6.3.1. The constant arising from S_2

Recall that

$$S_2^{\pm} = \sum_{\substack{\frac{y^B}{\log y} \le n \le y^B \\ P(n) \le y}} \chi(n) \frac{1 - e(\pm \alpha n)}{n} - \sum_{\substack{y^B \le n \le y^B (\log y)^5 \\ P(n) \le y}} \chi(n) \frac{e(\pm \alpha n)}{n}. \tag{6.3.1}$$

We show that that if χ pretends to be 1, then we have

Proposition 6.3.2. *Let $S_2^{\pm}$ be as in (6.3.1) with $y \ge 2$ and $1 \le B < \exp((\log y)^{3/5-\epsilon}$. Let χ be in $A_{\pm}(N, T)$, then*

$$S_2^{\pm} = \rho(B) \left(\gamma + \log(2\pi) \mp \frac{i\pi}{2} \right) + O \left(h(T) \rho(B-1) \log\log y + \frac{\rho(B) \log(B+1)(\log\log y)^2}{\log y} \right).$$

Proof. We start by using Proposition 4.1.2 with $f(n) = 1 - e(\pm\alpha n)$ for the first sum and $f(n) = e(\pm\alpha n)$ for the second sum to approximate χ by 1. We have

$$\begin{aligned}
\sum_{\substack{\frac{y^B}{\log y} \le n \le y^B \\ P(n) \le y}} \chi(n) \frac{1 - e(\pm \alpha n)}{n} - \sum_{\substack{y^B \le n \le y^B (\log y)^5 \\ P(n) \le y}} \chi(n) \frac{e(\pm \alpha n)}{n} &= \sum_{\substack{\frac{y^B}{\log y} \le n \le y^B \\ P(n) \le y}} \frac{1 - e(\pm \alpha n)}{n} - \sum_{\substack{y^B \le n \le y^B (\log y)^5 \\ P(n) \le y}} \frac{e(\pm \alpha n)}{n} \\
&\quad + O\left(h(y) \log y \int_{B-1-\frac{\log\log y}{\log y}}^{B-1+\frac{c \log\log y}{\log y}} \rho(u) du \right) \\
&= \sum_{\substack{\frac{y^B}{\log y} \le n \le y^B \\ P(n) \le y}} \frac{1 - e(\pm \alpha n)}{n} - \sum_{\substack{y^B \le n \le y^B (\log y)^5 \\ P(n) \le y}} \frac{e(\pm \alpha n)}{n} \\
&\quad + O\left(h(T) \rho(B-1) \log\log y \right),
\end{aligned}$$

where we bounded the integral with Lemma 1.3.5.

Next, to evaluate the right hand side, we start by removing the smoothness condition with Lemma 5.2.2 and then we throw back in the end ranges to the summations using Lemmas 5.1.3 and 5.1.2 in which we take $\alpha = \frac{1}{y^B}$ and $c = 5\left(1 - \frac{\log B}{\log|\log \alpha|}\right)$. This gives us

$$\sum_{\substack{\frac{y^B}{\log y}\le n\le y^B\\ P(n)\le y}} \chi(n)\frac{1-e(\pm\alpha n)}{n} - \sum_{\substack{y^B\le n\le y^B(\log y)^5\\ P(n)\le y}} \chi(n)\frac{e(\pm\alpha n)}{n} = \rho(B)\left(\sum_{\frac{y^B}{\log y}\le n\le y^B}\frac{1-e(\pm\alpha n)}{n} - \sum_{y^B\le n\le y^B(\log y)^5}\frac{e(\pm\alpha n)}{n}\right)$$

$$+O\left(h(T)\rho(B-1)\log\log y + \frac{\rho(B)\log(B+1)(\log\log y)^2}{\log y}\right)$$

$$=\rho(B)\left(\sum_{n\le y^B}\frac{1-e(\pm\alpha n)}{n} - \sum_{n\ge y^B}\frac{e(\pm\alpha n)}{n}\right)$$

$$+O\left(h(T)\rho(B-1)\log\log y + \frac{\rho(B)\log(B+1)(\log\log y)^2}{\log y}\right).$$

Finally, appealing to Lemma 5.1.4, we obtain

$$\sum_{\substack{\frac{y^B}{\log y}\le n\le y^B\\ P(n)\le y}} \chi(n)\frac{1-e(\pm\alpha n)}{n} - \sum_{\substack{y^B\le n\le y^B(\log y)^5\\ P(n)\le y}} \chi(n)\frac{e(\pm\alpha n)}{n} = \rho(B)\left(\gamma+\log(2\pi)-\mp\frac{i\pi}{2}\right)$$

$$+O\left(h(T)\rho(B-1)\log\log y + \frac{\rho(B)\log(B+1)(\log\log y)^2}{\log y}\right),$$

which proves the proposition. □

Proposition 6.3.1 and Proposition 6.3.2 give us the desired lower bound for Theorems 1 and 2 respectively, given that we can pick characters in $A_\pm(N,T)$. So we now find the 1-pretentious character we need, which will lead us to the culminating point of this thesis of putting everything together and proving Theorems 1 and 2.

6.3.2. Smooth 1-pretentious characters

Observe that deriving bounds for characters in $A_\pm(N,T)$ gives rise to $h(T)=\frac{\log\log T}{N}$ in the error terms and we now need to choose T and N in terms of y so that the size of the error terms does not get too large.

Now, we already know from Propositions 4.2.1 and 4.2.2 that there are many characters pretending to be 1. Indeed, we have that

$$|A_+(N,T)| \gg \frac{q}{N^{\frac{2T}{\log T}}}, \tag{6.3.2}$$

and that for all but at most $Q^{\frac{1}{10}}$ primes $q\le Q$,

$$|A_-(N,T)| \gg \frac{q}{N^{\frac{2T}{\log T}}}. \tag{6.3.3}$$

From now on, $A_-(N,T)$ will refer to the admissible prime moduli q for which the bound holds.

Now, recall that we have restricted our characters to be in the set A_δ defined as in (6.1.2), so we need to make sure that $A_\delta \cap A_\pm(N,T) \neq \emptyset$. That is, we need to choose T and N, so that $A_\pm(N,T)$ contains enough characters to ensure non-empty intersection with A_δ. This means that we need to choose the right balance between N and T, so we choose $T = \frac{y}{4\log y}$ and $N = \log y$, giving

$$h(T) = h(y) \ll \frac{\log\log y}{\log y}.$$

We show

Proposition 6.3.3. *Let $y = \log q$, and let $A_\pm = A_\pm(T,N)$ for $N = \log y$ and $T = \frac{y}{4\log y}$. Then*

$$|A_\pm| \gg q^{1-\frac{\log\log\log q}{(\log\log q)^2}}.$$

Proof. We know, as stated in (6.3.2) and (6.3.3), that

$$|A_\pm(N,T)| \gg \frac{q}{N^{\frac{2T}{\log T}}}.$$

Now, as $y = \log q$, and given our choice $N = \log y$ and $T = \frac{y}{4\log y}$, we have

$$\begin{aligned} N^{\frac{2T}{\log T}} &= \exp\left(\frac{2T}{\log T}(\log\log y)\right) \\ &\leq \exp\left(\frac{y}{2\log y}\frac{\log\log y}{\log T}\right) \\ &\leq \exp\left(\frac{y\log\log y}{\log^2 y}\right) \\ &= q^{\frac{\log\log y}{\log^2 y}}, \end{aligned}$$

from which, we deduce that

$$|A_\pm| \gg q^{1-\frac{\log\log\log q}{(\log\log q)^2}}.$$

□

Corollary 6.3.1. *Let A_δ, $A_\pm$ be the sets defined as above. If $\delta > \frac{(\log\log\log q)}{(\log\log q)^{\frac{1}{2}}}$, then*

$$|A_\delta \cap A_\pm| \gg q^{1-\frac{\log\log\log q}{(\log\log q)^2}}.$$

Proof. Let $\mathcal{A} = \{\chi \pmod q : \chi \notin A_\delta\}$ be the exceptional set of A_δ and suppose that $\delta > \frac{(\log\log\log q)}{(\log\log q)^{\frac{1}{2}}}$, then by (6.1.3) we have that

$$|\mathcal{A}| \ll q^{1-\left(\frac{\log\log\log q}{\log\log q}\right)^2}.$$

That is, using Proposition 6.3.3 we get

$$\begin{aligned}
|A_\delta \cap A_\pm| &= |A_\pm| - |A_\pm \cap \mathcal{A}| \\
&\geq |A_\pm| - |\mathcal{A}| \\
&\gg q^{1-\frac{\log\log\log q}{(\log\log q)^2}} - q^{1-\left(\frac{\log\log\log q}{\log\log q}\right)^2} \\
&\gg q^{1-\frac{\log\log\log q}{(\log\log q)^2}},
\end{aligned}$$

as claimed. $\square$

Now that we have found at least a character to work with, we finally have the ingredients we need and are ready to go forward with the proof of Theorem 1.

6.4. Proof of Theorem 1

We are now ready to prove our main theorem, along with Theorem 2.

Proof of Theorem 1. Let q be a prime for which the bounds (6.3.2) and (6.3.3) hold. Starting with Pólya's Fourier expansion, we have

$$\sum_{n \leq \alpha q} \chi(n) = \frac{\tau(\chi)}{2\pi i} \sum_{1 \leq |n| \leq z} \frac{\overline{\chi(n)}}{n} (1 - e(-\alpha n)) + O\left(\frac{q \log q}{z}\right),$$

where we let $z = q^{\frac{11}{21}}$.

Now we let $\delta = \frac{\log\log y}{(\log y)^{\frac{1}{2}}}$ in (6.1.2), so that by Corollary 6.3.1 $|A_\delta \cap A_\pm| \neq \emptyset$, and we choose a character χ in the intersection. Notice that this means that it holds for χ with $h(y) = \frac{\log\log y}{\log y}$. We have

$$\begin{aligned}
\sum_{1 \leq |n| \leq z} \frac{\chi(n)}{n} (1 - e(-\alpha n)) &= \sum_{1 \leq n \leq z} \frac{\chi(n)}{n} (1 - e(-\alpha n)) - \chi(-1) \sum_{1 \leq n \leq z} \frac{\chi(n)}{n} (1 - e(\alpha n)) \\
&= (S_1 + S_2^- + S_3^-) - \chi(-1)(S_1 + S_2^+ + S_3^+).
\end{aligned}$$

At this point we need to treat the odd and even character cases separately.

If χ is an even character, then we get cancellation of S_1 and we are left with a contribution from $S_2^{\pm}$ and an error term from $S_3^{\pm}$. Hence using Propositions 6.2.1 and 6.3.2, with $h(y) = \frac{\log\log y}{\log y}$, we get

$$\sum_{1 \le |n| \le z} \frac{\chi(n)}{n}(1 - e(-\alpha n)) = i\pi\rho(B) + O\left(\frac{\rho(B-1)(\log\log y)^2}{\log y}\right),$$

and thus, going back to (6.1.1), we obtain

$$\begin{aligned}\sum_{n \le \alpha q} \overline{\chi}(n) &= \frac{\tau(\chi)}{2\pi i}\left(i\pi\rho(B) + O\left(\frac{\rho(B-1)(\log\log y)^2}{\log y}\right)\right) + O(q^{10/21}\log q) \\ &= \frac{\tau(\chi)\rho(B)}{2} + O\left(\frac{\sqrt{q}\rho(B-1)(\log\log y)^2}{\log y}\right).\end{aligned}$$

Recalling that $y = \log q$ and that $|\tau(\chi)| = \sqrt{q}$, we get

$$\max_{\substack{\chi \ne \chi_0 \\ \chi \text{ even}}} \left|\sum_{n \le \frac{q}{(\log q)^B}} \chi(n)\right| \ge \frac{\rho(B)}{2}\sqrt{q} + O\left(\frac{\sqrt{q}\rho(B-1)(\log\log\log q)^2}{\log\log q}\right),$$

as desired.

Remark. *Note that the restriction on q is unnecessary for the even character case and that Theorem 2 holds for any prime q.*

As for the odd character case, given $\chi \in A_\delta \cap A_-$, we use Propositions 6.2.1 for S_3^+ and S_3^-, Proposition 6.3.2 for S_2^+ and S_2^- and Proposition 6.3.1 for S_1, to obtain

$$\begin{aligned}\sum_{1 \le |n| \le z} \frac{\chi(n)}{n}(1 - e(-\alpha n)) &= 2\log y \int_B^\infty \rho(u)du + 2\rho(B)(\gamma\log(2\pi)) + O(\log\log y) \\ &= 2\log y \int_B^\infty \rho(u)du + O(\log\log y),\end{aligned}$$

where the error term is arising from Proposition 4.1.3. As a consequence, using (6.1.1), we deduce that

$$\begin{aligned}\sum_{n\le \alpha q}\overline{\chi}(n) &= \frac{\tau(\chi)}{2\pi i}\left(2\log y\int_B^\infty \rho(u)du + O(\log\log y)\right)\\ &= \frac{\tau(\chi)}{\pi i}\log y\int_B^\infty \rho(u)du + O(\sqrt{q}\log\log y),\end{aligned}$$

from which we conclude that

$$\max_{\substack{\chi\neq\chi_0\\ \chi\text{ odd}}}\left|\sum_{n\le\frac{q}{(\log q)^B}}\chi(n)\right| \ge \frac{\sqrt{q}}{\pi}\log\log q\int_B^\infty \rho(u)du + O(\sqrt{q}\log\log\log q),$$

thus proving the theorem. $\square$

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